

Find the general solution for the third-order ODE  $x^2y''' + 4xy'' + 2y' = \ln x$ . [101 交大機械丙 5]

[解]原式 $\Rightarrow x^3y''' + 4x^2y'' + 2xy' = x \ln x \cdots \cdots \cdots$ (i), 令  $x = e^t \Rightarrow t = \ln x$

$$(i) \Rightarrow \frac{d}{dt} \left( \frac{d}{dt} - 1 \right) \left( \frac{d}{dt} - 2 \right) y + 4 \frac{d}{dt} \left( \frac{d}{dt} - 1 \right) y + 2 \frac{dy}{dt} = te^t \Rightarrow \frac{d^3 y}{dt^3} + \frac{d^2 y}{dt^2} = te^t \cdots \cdots \cdots (ii)$$

特徵方程式  $\lambda^3 + \lambda^2 = 0 \Rightarrow \lambda = 0, 0, -1 \Rightarrow y_h = C_1 + C_2 t + C_3 e^{-t}$

令  $y_p = (At + B)e^t \Rightarrow y_p' = [At + (A + B)]e^t \Rightarrow y_p'' = [At + (2A + B)]e^t \Rightarrow y_p''' = [At + (3A + B)]e^t$

(ii)  $\Rightarrow [At + (3A + B)]e^t + [At + (2A + B)]e^t = te^t \Rightarrow [2At + (5A + 2B)]e^t = te^t$

$$A = \frac{1}{2}, B = -\frac{5}{4}$$

$$y = y_h + y_p = C_1 + C_2 t + C_3 e^{-t} + \left( \frac{1}{2} t - \frac{5}{4} \right) e^t = C_1 + C_2 \ln x + C_3 x^{-1} + \left( \frac{1}{2} \ln x - \frac{5}{4} \right) x$$