(a)Find the Fourier series of the function $f(x)$, where $f(x)=x^2$，$-\pi \le x \le \pi$. (b)Use the results in (a) to prove $\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$. (c)Use the results in (a) to calculate $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=?$ [98 彰師大機電 4]

[解](a)令 $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$，因 $f(x)$為偶函數 $b_n=0$

$$a_0=\frac{2}{\pi}\int_0^{\pi}x^2dx=\frac{2\pi^2}{3}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}x^2\cos nxdx=\frac{2}{n\pi}(x^2\sin nx\Big|_0^{\pi}-2\int_0^{\pi}x\sin nxdx)$$

$$=\frac{4}{n^2\pi}(x\cos nx\Big|_0^{\pi}-\int_0^{\pi}\cos nxdx)=(-1)^n\frac{4}{n^2}$$

$$x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

(b)$x=\pi$代入(a)得

$$\frac{f(\pi^-)+f(\pi^+)}{2}=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n\Rightarrow\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

(c)$x=0$代入(a)得

$$\frac{f(0^-)+f(0^+)}{2}=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\Rightarrow 0=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$$