If $f(t) = \sin \pi t$ for $t \in (-\pi, \pi]$ be a function of period 2π. Find the Fourier Series representation of $f(t)$. [98 台灣聯大 C]

[解] $f(t)$為奇函數 $\Rightarrow$ 設 $f(t) = \sum_{n=1}^{\infty} b_n \sin nt$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{2}{\pi}\int_{0}^{\pi} \sin \pi t \sin nt\,dt = \frac{1}{\pi}\int_{0}^{\pi} [\cos(n-\pi)t - \cos(n+\pi)t]dt$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-\pi)t}{n-\pi} - \frac{\sin(n+\pi)t}{n+\pi}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin(n-\pi)\pi}{n-\pi} - \frac{\sin(n+\pi)\pi}{n+\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin n\pi \cos \pi^2 - \cos n\pi \sin \pi^2}{n-\pi} - \frac{\sin n\pi \cos \pi^2 + \cos n\pi \sin \pi^2}{n+\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{0\cdot \cos \pi^2 - (-1)^n \sin \pi^2}{n-\pi} - \frac{0\cdot \cos \pi^2 + (-1)^n \sin \pi^2}{n+\pi}\right] = -\frac{(-1)^n \sin \pi^2}{\pi}\left(\frac{1}{n+\pi} + \frac{1}{n-\pi}\right)$$

$$= -\frac{(-1)^n \sin \pi^2}{\pi}\left[\frac{(n-\pi)+(n+\pi)}{n^2-\pi^2}\right] = -\frac{2n(-1)^n \sin \pi^2}{\pi(n^2-\pi^2)}$$

$$f(t) = \sum_{n=1}^{\infty} -\frac{2n(-1)^n \sin \pi^2}{\pi(n^2-\pi^2)}\sin nt = -\frac{2\sin \pi^2}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{n^2-\pi^2}\sin nt$$