令 $f(x)=\dfrac{x^2}{2}$ ，$-\pi \le x \le \pi$，試以傅立葉級數展開並以此求級數 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ 。[97 虎尾機械 4]

[解]令 $f(x)=\dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$$a_0=\frac{2}{\pi}\int_0^{\pi}\frac{x^2}{2}dx=\frac{1}{\pi}\left.\frac{x^3}{3}\right|_0^{\pi}=\frac{\pi^2}{3}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}\frac{x^2}{2}\cos nx dx=\frac{1}{n\pi}\left(x^2\sin nx\Big|_0^{\pi}-2\int_0^{\pi}x\sin nx dx\right)$$

$$=\frac{2}{n^2\pi}\left(x\cos nx\Big|_0^{\pi}-\int_0^{\pi}\cos nx dx\right)=(-1)^n\frac{2}{n^2}$$

$$b_n=0$$

$$f(x)=\frac{\pi^2}{6}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

令 $x=\pi$ 代入

$$\frac{f(\pi^-)+f(\pi^+)}{2}=\frac{\pi^2}{6}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n \Rightarrow \frac{\frac{\pi^2}{2}+\frac{\pi^2}{2}}{2}=\frac{\pi^2}{6}+2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{3}=2\sum_{n=1}^{\infty}\frac{1}{n^2}\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$