

Show that the Fourier series of $f(x) = x, -\pi < x < \pi$ leads to $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$. [97 中央機械能源光機電生醫 8]

[解] $f(x)$ 為奇函數 \Rightarrow 令 $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin nx dx = -\frac{2}{n\pi} (x \cos nx \Big|_0^{\pi} - \int_0^{\pi} \cos nx dx) = -\frac{2}{n\pi} (\pi \cos n\pi) = \frac{2(-1)^{n-1}}{n}$$

$$f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin nx, \quad x = \frac{\pi}{2} \text{ 代入得 } \frac{f(\frac{\pi^-}{2}) + f(\frac{\pi^+}{2})}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin \frac{n\pi}{2}$$

$$\frac{\pi}{2} = 2(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots) \Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$