Please use Fourier integral representation to show that $\int_0^{\infty} \frac{\cos x\omega + \omega \sin x\omega}{1+\omega^2} d\omega = \begin{cases} 0, & x<0 \\ \frac{\pi}{2}, & x=0 \\ \pi e^{-x}, & x>0 \end{cases}$.

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[解]令 $f(x) = \begin{cases} 0, & x<0 \\ e^{-x}, & x>0 \end{cases} \Rightarrow$ 設 $f(x) = \int_0^{\infty} [a(\omega)\cos\omega x + b(\omega)\sin\omega x] d\omega$

先推導 $\int e^{ax} e^{ibx} dx = \int e^{(a+ib)x} dx = \frac{e^{(a+ib)x}}{a+ib} = \frac{(a-ib)e^{(a+ib)x}}{a^2+b^2} = \frac{e^{ax}(a-ib)(\cos bx + i\sin bx)}{a^2+b^2}$

$$= \frac{e^{ax}[(a\cos bx + b\sin bx) + i(a\sin bx - b\cos bx)]}{a^2+b^2}$$

實部 $\int e^{ax}\cos bx dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$, 虛部 $\int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

$$a(\omega) = \frac{1}{\pi}\int_0^{\infty} f(x)\cos\omega x dx = \frac{1}{\pi}\int_0^{\infty} e^{-x}\cos\omega x dx = \frac{1}{\pi}\cdot\left.\frac{e^{-x}(-\cos\omega x + \omega\sin\omega x)}{1+\omega^2}\right|_0^{\infty}$$

$$= \frac{0+1}{\pi(1+\omega^2)} = \frac{1}{\pi(1+\omega^2)}$$

$$b(\omega) = \frac{1}{\pi}\int_0^{\infty} f(x)\sin\omega x dx = \frac{1}{\pi}\int_0^{\infty} e^{-x}\sin\omega x dx = \frac{1}{\pi}\cdot\left.\frac{e^{-x}(-\sin\omega x - \omega\cos\omega x)}{1+\omega^2}\right|_0^{\infty}$$

$$= \frac{0+\omega}{\pi(1+\omega^2)} = \frac{\omega}{\pi(1+\omega^2)}$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} d\omega$$

當 $x<0$ 時，$\frac{f(x^-)+f(x^+)}{2} = \frac{1}{\pi}\int_0^{\infty}\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}d\omega \Rightarrow \frac{0+0}{2} = \frac{1}{\pi}\int_0^{\infty}\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}d\omega$

$$\int_0^{\infty}\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}d\omega = 0$$

當 $x=0$ 時，$\frac{f(0^-)+f(0^+)}{2} = \frac{1}{\pi}\int_0^{\infty}\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}d\omega \Rightarrow \frac{0+1}{2} = \frac{1}{\pi}\int_0^{\infty}\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}d\omega$

$$\int_0^{\infty}\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}d\omega = \frac{\pi}{2}$$

當 $x>0$ 時，$\frac{f(x^-)+f(x^+)}{2} = \frac{1}{\pi}\int_0^{\infty}\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}d\omega$

$$\frac{e^{-x}+e^{-x}}{2} = \frac{1}{\pi}\int_0^{\infty}\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}d\omega \Rightarrow \int_0^{\infty}\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}d\omega = \pi e^{-x}$$

[註]原題目誤植 $\int_0^{\infty}\frac{\cos x\omega + \omega\sin x\omega}{1+\omega^2}d\omega = \begin{cases} 0, & x<0 \\ \frac{\pi}{2}, & x=0 \\ xe^{-x}, & x>0 \end{cases}$ ，本題已修改。