(a)Expand $f(x) = x + \pi$, $-\pi < x < \pi$ in a Fourier series. (b)Use the result of (a) to find

$1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$. [94 中央機械 7]

[解](a)令 $F(x) = f(x) - \pi$，則 $F(x)$為奇函數，其 Fourier 級數為

$F(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin nx$，其中

$b_n = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} x \sin nx \, dx = \dfrac{2}{\pi}(-\dfrac{1}{n})\left(x \cos nx \Big|_0^{\pi} - \int_0^{\pi} \cos nx \, dx \right)$

$\quad = -\dfrac{2}{n\pi}(\pi \cos n\pi) = -\dfrac{2}{n}(-1)^n$

$F(x) = -2\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} \sin nx \Rightarrow f(x) = \pi - 2\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} \sin nx$

(b)令 $x = \dfrac{\pi}{2}$ 代入

$\dfrac{f(\frac{\pi}{2}^-) + f(\frac{\pi}{2}^+)}{2} = \pi - 2\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} \sin \dfrac{n\pi}{2} \Rightarrow \dfrac{\frac{3\pi}{2} + \frac{2\pi}{2}}{2} = \pi - 2(-\dfrac{1}{1} + \dfrac{1}{3} - \dfrac{1}{5} + \dfrac{1}{7} - \cdots)$

$\dfrac{3\pi}{2} = \pi + 2(\dfrac{1}{1} - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots) \Rightarrow \dfrac{1}{1} - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots = \dfrac{\pi}{4}$