

(a) Find the Fourier series of periodic function $f(x) = \begin{cases} k, & \text{if } -\pi/2 < x < \pi/2 \\ 0, & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$. (b) Show

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}. \quad [91 \text{ 清大動機 } 5]$$

[解] 令 $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{\pi} \int_{-\pi/2}^{3\pi/2} f(x) dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} k \cdot dx = \frac{k}{\pi} \cdot x \Big|_{-\pi/2}^{\pi/2} = k$$

$$a_n = \frac{1}{\pi} \int_{-\pi/2}^{3\pi/2} f(x) \cos nxdx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} k \cos nxdx = \frac{k}{n\pi} \cdot \sin nx \Big|_{-\pi/2}^{\pi/2} = \frac{k}{n\pi} \left[\sin \frac{n\pi}{2} - \sin \left(-\frac{n\pi}{2} \right) \right]$$

$$= \frac{2k}{n\pi} \sin \frac{n\pi}{2} = \begin{cases} 0, & n \text{ 為偶數} \\ \frac{2k}{n\pi}, & n = 1, 5, 9, \dots = \frac{2k(-1)^{n-1}}{(2n-1)\pi} \\ -\frac{2k}{n\pi}, & n = 3, 7, 11, \dots \end{cases}$$

$$b_n = \frac{1}{\pi} \int_{-\pi/2}^{3\pi/2} f(x) \sin nxdx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} k \sin nxdx = -\frac{k}{n\pi} \cdot \cos nx \Big|_{-\pi/2}^{\pi/2} = 0$$

$$f(x) = \frac{k}{2} + \frac{2k}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos(2n-1)x$$

令 $x=0$ 代入

$$\frac{f(0^-) + f(0^+)}{2} = \frac{k}{2} + \frac{2k}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \Rightarrow \frac{k+k}{2} = \frac{k}{2} + \frac{2k}{\pi} \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right)$$

$$\frac{k}{2} = \frac{2k}{\pi} \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right) \Rightarrow \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$