Find the Fourier series of the function $f(x)=\begin{cases} x, & 0\le x<1 \\ 0, & 1\le x<2 \end{cases}$. [91 成大造船 2]

[解]設 $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\pi x+b_n\sin n\pi x)$, 其中

$$a_0=\int_0^2 f(x)dx=\int_0^1 xdx+\int_1^2 0dx=\left.\frac{x^2}{2}\right|_0^1=\frac{1}{2}$$

$$a_n=\int_0^2 f(x)\cos n\pi x dx=\int_0^1 x\cos n\pi x dx=\frac{1}{n\pi}\left(x\sin n\pi x\Big|_0^1-\int_0^1\sin n\pi x dx\right)$$

$$=\left.\frac{\cos n\pi x}{n^2\pi^2}\right|_0^1=\frac{\cos n\pi-1}{n^2\pi^2}=\frac{(-1)^n-1}{n^2\pi^2}$$

$$b_n=\int_0^2 f(x)\sin n\pi x dx=\int_0^1 x\sin n\pi x dx=-\frac{1}{n\pi}\left(x\cos n\pi x\Big|_0^1-\int_0^1\cos n\pi x dx\right)$$

$$=-\frac{\cos n\pi}{n\pi}=-\frac{(-1)^n}{n\pi}$$

$$f(x)=\frac{1}{4}+\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2\pi^2}\cos n\pi x-\frac{(-1)^n}{n\pi}\sin n\pi x\right],\quad 0\le x<2$$