Show that the Fourier transformation of $e^{-\alpha t^2}$ is $\sqrt{\frac{\pi}{\alpha}}e^{-\frac{\omega^2}{4\alpha}}$. [106 台大機械 5(a)]

[解]$\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} e^{-\alpha t^2}e^{-i\omega t}dt = \int_{-\infty}^{\infty} e^{-\alpha(t^2+\frac{i\omega}{\alpha}t)}dx = \int_{-\infty}^{\infty} e^{-\alpha(t+\frac{i\omega}{2\alpha})^2-\frac{\omega^2}{4\alpha}}dt$

$$= e^{-\frac{\omega^2}{4\alpha}}\int_{-\infty}^{\infty} e^{-\alpha(t+\frac{i\omega}{2\alpha})^2}dt$$

令 $u = t + \frac{i\omega}{2\alpha}$，上式為

$$e^{-\frac{\omega^2}{4\alpha}}\int_{-\infty}^{\infty} e^{-\alpha u^2}du = e^{-\frac{\omega^2}{4\alpha}}\int_{-\infty}^{\infty} e^{-(\sqrt{\alpha}u)^2}du = e^{-\frac{\omega^2}{4\alpha}}\cdot\frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-(\sqrt{\alpha}u)^2}d(\sqrt{\alpha}u) = e^{-\frac{\omega^2}{4\alpha}}\cdot\sqrt{\frac{\pi}{\alpha}}$$