If $r(x)=x^2$, $0<x<2\pi$, $r(x)=r(x+2\pi)$. (a)Find $r(x)$ in the Fourier series. (b)Evaluate $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

(c)Evaluate $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. (d)Evaluate $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$. [104 中央機械甲乙丙能源光機電乙 2]

[解](a)令$r(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}x^2dx=\frac{1}{\pi}\cdot\frac{x^3}{3}\Big|_0^{2\pi}=\frac{8\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x^2\cos nxdx=\frac{1}{n\pi}(x^2\sin nx\Big|_0^{2\pi}-2\int_0^{2\pi}x\sin nxdx)$$

$$=\frac{2}{n^2\pi}(x\cos nx\Big|_0^{2\pi}-\int_0^{2\pi}\cos nxdx)=\frac{4}{n^2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}x^2\sin nxdx=-\frac{1}{n\pi}(x^2\cos nx\Big|_0^{2\pi}-2\int_0^{2\pi}x\cos nxdx)$$

$$=-\frac{4\pi}{n}+\frac{2}{n\pi}(x\sin nx\Big|_0^{2\pi}-\int_0^{2\pi}\sin nxdx)=-\frac{4\pi}{n}$$

$$r(x)=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}(\frac{1}{n^2}\cos nx-\frac{\pi}{n}\sin nx)$$

(b)令 $x=2\pi$ 代入(a)得

$$\frac{r(2\pi^-)+r(2\pi^+)}{2}=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 2n\pi\Rightarrow\frac{4\pi^2+0}{2}=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{2\pi^2}{3}=4\sum_{n=1}^{\infty}\frac{1}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}\cdots\cdots\cdots\text{(i)}$$

(c)令 $x=\pi$ 代入(a)得

$$\frac{r(\pi^-)+r(\pi^+)}{2}=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi\Rightarrow\frac{\pi^2+\pi^2}{2}=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{3}=4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=-\frac{\pi^2}{12}\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}\cdots\cdots\cdots\text{(ii)}$$

(d)將(i)式與(ii)式相加得

$$2\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{6}+\frac{\pi^2}{12}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$$