

Please use the Fourier integral to show that  $\int_0^{\infty} \frac{\sin \pi \omega \sin x \omega}{1 - \omega^2} d\omega = \begin{cases} \frac{\pi}{2} \sin x, & \text{if } 0 \leq x \leq \pi \\ 0, & \text{if } x > \pi \end{cases}$ . [101 暨

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[解] 設  $f(x) = \begin{cases} \sin x, & 0 \leq x \leq \pi \\ 0, & x > \pi \end{cases} \Rightarrow \text{令 } f(x) = \int_0^{\infty} b(\omega) \sin \omega x d\omega$

$$\begin{aligned} b(\omega) &= \frac{2}{\pi} \int_0^{\infty} f(x) \sin \omega x dx = \frac{2}{\pi} \int_0^{\pi} \sin x \sin \omega x dx \\ &= \frac{1}{\pi} \int_0^{\pi} \cos(1 - \omega)x - \cos(1 + \omega)x dx = \frac{1}{\pi} \cdot \left[ \frac{\sin(1 - \omega)x}{1 - \omega} - \frac{\sin(1 + \omega)x}{1 + \omega} \right] \Big|_0^{\pi} \\ &= \frac{1}{\pi} \left[ \frac{\sin(1 - \omega)\pi}{1 - \omega} - \frac{\sin(1 + \omega)\pi}{1 + \omega} \right] \\ &= \frac{1}{\pi} \left( \frac{\sin \pi \cos \omega \pi - \cos \pi \sin \omega \pi}{1 - \omega} - \frac{\sin \pi \cos \omega \pi + \cos \pi \sin \omega \pi}{1 + \omega} \right) = \frac{1}{\pi} \left( \frac{\sin \omega \pi}{1 - \omega} - \frac{-\sin \omega \pi}{1 + \omega} \right) \\ &= \frac{\sin \omega \pi}{\pi} \left( \frac{1}{1 - \omega} + \frac{1}{1 + \omega} \right) = \frac{\sin \omega \pi}{\pi} \frac{2}{1 - \omega^2} \end{aligned}$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \omega \pi}{1 - \omega^2} \sin \omega x d\omega$$

當  $0 < x < \pi$  時,  $\frac{f(x^-) + f(x^+)}{2} = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \omega \pi}{1 - \omega^2} \sin \omega x d\omega$

$$\frac{\sin x + \sin x}{2} = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \omega \pi}{1 - \omega^2} \sin \omega x d\omega \Rightarrow \int_0^{\infty} \frac{\sin \omega \pi}{1 - \omega^2} \sin \omega x d\omega = \frac{\pi}{2} \sin x$$

當  $x = 0, \pi$  時,  $\frac{f(x^-) + f(x^+)}{2} = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \omega \pi}{1 - \omega^2} \sin \omega x d\omega \Rightarrow \frac{0 + 0}{2} = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \omega \pi}{1 - \omega^2} \sin \omega x d\omega$

$$\int_0^{\infty} \frac{\sin \omega \pi}{1 - \omega^2} \sin \omega x d\omega = 0$$

當  $x > \pi$  時,  $\frac{f(x^-) + f(x^+)}{2} = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \omega \pi}{1 - \omega^2} \sin \omega x d\omega \Rightarrow \frac{0 + 0}{2} = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \omega \pi}{1 - \omega^2} \sin \omega x d\omega$

$$\int_0^{\infty} \frac{\sin \omega \pi}{1 - \omega^2} \sin \omega x d\omega = 0$$

因此  $\int_0^{\infty} \frac{\sin \omega \pi}{1 - \omega^2} \sin \omega x d\omega = \begin{cases} \frac{\pi}{2} \sin x, & 0 \leq x \leq \pi \\ 0, & x > \pi \end{cases}$