Use the Fourier series of the function $f(x)=\begin{cases}0, & \text{if } 0<x<\pi \\ 1, & \text{if } \pi<x<2\pi\end{cases}$, find the sum $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$.

[101 宜蘭電機 5]

[解]令 $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)dx=\frac{1}{\pi}\int_\pi^{2\pi} 1\cdot dx=\frac{1}{\pi}\cdot x\Big|_\pi^{2\pi}=1$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx=\frac{1}{\pi}\int_\pi^{2\pi} 1\cdot\cos nxdx=\frac{1}{n\pi}\cdot\sin nx\Big|_\pi^{2\pi}=0$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx=\frac{1}{\pi}\int_\pi^{2\pi} 1\cdot\sin nxdx=-\frac{1}{n\pi}\cdot\cos nx\Big|_\pi^{2\pi}=-\frac{1}{n\pi}\cdot(\cos 2n\pi-\cos n\pi)$$

$$=-\frac{1}{n\pi}\cdot[1-(-1)^n]=-\frac{2}{(2n-1)\pi}$$

$$f(x)=\frac{1}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$$

令 $x=\frac{\pi}{2}$ 代入

$$\frac{f(\frac{\pi}{2}^-)+f(\frac{\pi}{2}^+)}{2}=\frac{1}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi}{2}\Rightarrow\frac{0+0}{2}=\frac{1}{2}-\frac{2}{\pi}(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots)$$

$$\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$$