Expand $f(x)=x^2$ for $0<x<L$, in a Fourier series. [100清大動機7(c)]

[解] $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{2n\pi x}{L}+b_n\sin\frac{2n\pi x}{L})$

$$a_0=\frac{2}{L}\int_0^L f(x)dx=\frac{2}{L}\int_0^L x^2dx=\frac{2}{L}\cdot\frac{x^3}{3}\Big|_0^L=\frac{2}{L}\cdot\frac{L^3}{3}=\frac{2L^2}{3}$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx=\frac{2}{L}\int_0^L x^2\cos\frac{2n\pi x}{L}dx$$

$$=\frac{2}{L}\cdot\frac{L}{2n\pi}(x^2\sin\frac{2n\pi x}{L}\Big|_0^L-2\int_0^L x\sin\frac{2n\pi x}{L}dx)$$

$$=\frac{1}{n\pi}[2\cdot\frac{L}{2n\pi}(x\cos\frac{2n\pi x}{L}\Big|_0^L-\int_0^L\cos\frac{2n\pi x}{L}dx)]=\frac{1}{n\pi}(\frac{L}{n\pi}\cdot L)=\frac{L^2}{n^2\pi^2}$$

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx=\frac{2}{L}\int_0^L x^2\sin\frac{2n\pi x}{L}dx$$

$$=-\frac{2}{L}\cdot\frac{L}{2n\pi}(x^2\cos\frac{2n\pi x}{L}\Big|_0^L-2\int_0^L x\cos\frac{2n\pi x}{L}dx)$$

$$=-\frac{1}{n\pi}[L^2-2\cdot\frac{L}{2n\pi}(x\sin\frac{2n\pi x}{L}\Big|_0^L-\int_0^L\sin\frac{2n\pi x}{L}dx)]$$

$$=-\frac{1}{n\pi}[L^2+\frac{L}{n\pi}\int_0^L\sin\frac{2n\pi x}{L}dx]=-\frac{1}{n\pi}[L^2-\frac{L}{n\pi}\cdot\frac{L}{2n\pi}\cos\frac{2n\pi x}{L}\Big|_0^L]=-\frac{L^2}{n\pi}$$

$$f(x)=\frac{L^2}{3}+\sum_{n=1}^{\infty}(\frac{L^2}{n^2\pi^2}\cos\frac{2n\pi x}{L}-\frac{L^2}{n\pi}\sin\frac{2n\pi x}{L}),\quad 0<x<L$$