Suppose a periodic function $f(t)$ with period is defined as $f(t)=\begin{cases}\frac{1}{k}, & 0\le t\le k\\ 0, & k\le t<2\end{cases}$, where k is a constant ($0<k<2$). Please expand $f(t)$ in a Fourier series. [100中原機械甲6]

[解] $f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\pi t+b_n\sin n\pi t)$

$$a_0=\int_0^2 f(t)dt=\int_0^k\frac{1}{k}dt=\frac{t}{k}\Big|_0^k=1$$

$$a_n=\int_0^2 f(t)\cos n\pi t dt=\int_0^k\frac{1}{k}\cos n\pi t dt=\frac{\sin n\pi t}{kn\pi}\Big|_0^k=\frac{\sin kn\pi}{kn\pi}$$

$$b_n=\int_0^2 f(t)\sin n\pi t dt=\int_0^k\frac{1}{k}\sin n\pi t dt=-\frac{\cos n\pi t}{kn\pi}\Big|_0^k=-\frac{\cos kn\pi-1}{kn\pi}$$

$$f(t)=\frac{1}{2}+\frac{1}{k\pi}\sum_{n=1}^{\infty}[\frac{\sin kn\pi}{n}\cos n\pi t-\frac{\cos kn\pi-1}{n}\sin n\pi t]$$