

$f(x) = x^2$  is defined within  $0 < x < 2\pi$ , and  $f(x)$  has a period  $2\pi$ , then (a) find the Fourier series of  $f(x)$ , (b) evaluate the sum of the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  from result of (a), (c) evaluate the sum of the series

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$  from result of (a). [99 交大機械甲 6]

[解](a) 令  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} x^2 dx = \frac{1}{\pi} \cdot \frac{x^3}{3} \Big|_0^{2\pi} = \frac{8\pi^2}{3}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nxdx = \frac{1}{n\pi} (x^2 \sin nx \Big|_0^{2\pi} - 2 \int_0^{2\pi} x \sin nxdx) \\ &= \frac{2}{n^2\pi} (x \cos nx \Big|_0^{2\pi} - \int_0^{2\pi} \cos nxdx) = \frac{4}{n^2} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nxdx = -\frac{1}{n\pi} (x^2 \cos nx \Big|_0^{2\pi} - 2 \int_0^{2\pi} x \cos nxdx) \\ &= -\frac{4\pi}{n} + \frac{2}{n\pi} (x \sin nx \Big|_0^{2\pi} - \int_0^{2\pi} \sin nxdx) = -\frac{4\pi}{n} \end{aligned}$$

$$f(x) = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \left( \frac{1}{n^2} \cos nx - \frac{\pi}{n} \sin nx \right)$$

(b) 令  $x = 2\pi$  代入(a)得

$$\frac{f(2\pi^-) + f(2\pi^+)}{2} = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos 2n\pi \Rightarrow \frac{4\pi^2 + 0}{2} = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(c) 令  $x = \pi$  代入(a)得

$$\frac{f(\pi^-) + f(\pi^+)}{2} = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi \Rightarrow \frac{\pi^2 + \pi^2}{2} = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$