

If $f(t) = \begin{cases} 2, & -2 \leq t < -1 \\ 1, & -1 \leq t < 1 \\ 2, & 1 \leq t < 2 \end{cases}$, (a) find the Fourier series of $f(t)$, (b) find the value of the Fourier series,

found in (a), converges to, when t is an integer, (c) find the steady state solution of the O.D.E.: $y'' + 25y = f(t)$, where $y'' = d^2y/dt^2$. [98 成大機械 5]

[解](a) $f(t)$ 為偶函數，其 Fourier 級數為 $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{2}$ ，其中

$$a_0 = \int_0^2 f(t) dt = \int_0^1 1 dt + \int_1^2 2 dt = t \Big|_0^1 + 2t \Big|_1^2 = 3$$

$$\begin{aligned} a_n &= \int_0^2 f(t) \cos \frac{n\pi t}{2} dt = \int_0^1 1 \cos \frac{n\pi t}{2} dt + \int_1^2 2 \cos \frac{n\pi t}{2} dt \\ &= \frac{2}{n\pi} \sin \frac{n\pi t}{2} \Big|_0^1 + \frac{4}{n\pi} \sin \frac{n\pi t}{2} \Big|_1^2 = \frac{2}{n\pi} \sin \frac{n\pi}{2} - \frac{4}{n\pi} \sin \frac{n\pi}{2} = -\frac{2}{n\pi} \sin \frac{n\pi}{2} = \frac{2(-1)^n}{(2n-1)\pi} \end{aligned}$$

$$f(t) = \frac{3}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \cos \frac{(2n-1)\pi t}{2}$$

(b) $t = -2$: 級數值 $\frac{f(-2^-) + f(-2^+)}{2} = \frac{2+2}{2} = 2$, $t = -1$: 級數值 $\frac{f(-1^-) + f(-1^+)}{2} = \frac{2+1}{2} = \frac{3}{2}$

$t = 0$: 級數值 $\frac{f(0^-) + f(0^+)}{2} = \frac{1+1}{2} = 1$, $t = 1$: 級數值 $\frac{f(1^-) + f(1^+)}{2} = \frac{1+2}{2} = \frac{3}{2}$

$$f(t+4) = f(t)$$

(c) $y'' + 25y = f(t)$ 的齊性解為 $y_h = C_1 \cos 5t + C_2 \sin 5t$

$$\text{令 } y_p = A \cos \frac{(2n-1)\pi t}{2} + B \sin \frac{(2n-1)\pi t}{2} + C$$

$$y_p' = \frac{(2n-1)\pi}{2} [-A \sin \frac{(2n-1)\pi t}{2} + B \cos \frac{(2n-1)\pi t}{2}]$$

$$y_p'' = \frac{(2n-1)^2 \pi^2}{4} [-A \cos \frac{(2n-1)\pi t}{2} - B \sin \frac{(2n-1)\pi t}{2}]$$

$$\text{代入方程式} \Rightarrow \frac{(2n-1)^2 \pi^2}{4} [-A \cos \frac{(2n-1)\pi t}{2} - B \sin \frac{(2n-1)\pi t}{2}]$$

$$+ 25[A \cos \frac{(2n-1)\pi t}{2} + B \sin \frac{(2n-1)\pi t}{2} + C] = \frac{3}{2} + \frac{2}{\pi} \cdot \frac{(-1)^n}{2n-1} \cos \frac{(2n-1)\pi t}{2}$$

$$25 - \frac{(2n-1)^2 \pi^2}{4} A = \frac{2}{\pi} \cdot \frac{(-1)^n}{2n-1}, B = 0, 25C = \frac{3}{2}$$

$$A = \frac{8(-1)^n}{\pi(2n-1)[100 - (2n-1)^2 \pi^2]}, B = 0, C = \frac{3}{50}$$

$$y = y_h + y_p = C_1 \cos 5t + C_2 \sin 5t + \frac{3}{50} + \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)[100 - (2n-1)^2 \pi^2]} \cos \frac{(2n-1)\pi t}{2}$$