If $f(t)=\sin\pi t$ for $t\in(-\pi,\pi]$ be a function of period 2π. Find the Fourier Series representation of $f(t)$. [98 台灣聯大 C]

[解] $f(t)$為奇函數 $\Rightarrow$ 設 $f(t)=\sum_{n=1}^{\infty} b_n \sin nt$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt=\frac{2}{\pi}\int_0^{\pi}\sin\pi t\sin nt\,dt=\frac{1}{\pi}\int_0^{\pi}[\cos(n-\pi)t-\cos(n+\pi)t]dt$$

$$=\frac{1}{\pi}\left[\frac{\sin(n-\pi)t}{n-\pi}-\frac{\sin(n+\pi)t}{n+\pi}\right]_0^{\pi}=\frac{1}{\pi}\left[\frac{\sin(n-\pi)\pi}{n-\pi}-\frac{\sin(n+\pi)\pi}{n+\pi}\right]$$

$$=\frac{1}{\pi}\left[\frac{\sin n\pi\cos\pi^2-\cos n\pi\sin\pi^2}{n-\pi}-\frac{\sin n\pi\cos\pi^2+\cos n\pi\sin\pi^2}{n+\pi}\right]$$

$$=\frac{1}{\pi}\left[\frac{0\cdot\cos\pi^2-(-1)^n\sin\pi^2}{n-\pi}-\frac{0\cdot\cos\pi^2+(-1)^n\sin\pi^2}{n+\pi}\right]=-\frac{(-1)^n\sin\pi^2}{\pi}\left(\frac{1}{n+\pi}+\frac{1}{n-\pi}\right)$$

$$=-\frac{(-1)^n\sin\pi^2}{\pi}\left[\frac{(n-\pi)+(n+\pi)}{n^2-\pi^2}\right]=-\frac{2n(-1)^n\sin\pi^2}{\pi(n^2-\pi^2)}$$

$$f(t)=\sum_{n=1}^{\infty}-\frac{2n(-1)^n\sin\pi^2}{\pi(n^2-\pi^2)}\sin nt=-\frac{2\sin\pi^2}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{n^2-\pi^2}\sin nt$$