

Find the Fourier series of $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \end{cases}$. [94中央機械能源8]

$$[\text{解}] f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{2} + b_n \sin \frac{n\pi x}{2} \right)$$

$$a_0 = \frac{1}{2} \int_{-2}^2 f(x) dx = \frac{1}{2} \left[\int_0^1 x dx + \int_1^2 1 dx \right] = \frac{1}{2} \left(\frac{x^2}{2} \Big|_0^1 + x \Big|_1^2 \right) = \frac{1}{2} \left(\frac{1}{2} + 1 \right) = \frac{3}{4}$$

$$\begin{aligned} a_n &= \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx = \frac{1}{2} \left[\int_0^1 x \cos \frac{n\pi x}{2} dx + \int_1^2 1 \cdot \cos \frac{n\pi x}{2} dx \right] \\ &= \frac{1}{2} \left[\frac{2}{n\pi} \left(x \sin \frac{n\pi x}{2} \Big|_0^1 - \int_0^1 \sin \frac{n\pi x}{2} dx \right) + \frac{2}{n\pi} \sin \frac{n\pi x}{2} \Big|_1^2 \right] \\ &= \frac{1}{2} \left[\frac{2}{n\pi} \left(1 + \frac{2}{n\pi} \cos \frac{n\pi x}{2} \Big|_0^1 \right) + \frac{2}{n\pi} (-\sin \frac{n\pi}{2}) \right] = \frac{1}{2} \left[\frac{2}{n\pi} \left(1 - \frac{2}{n\pi} \right) - \frac{2}{n\pi} \sin \frac{n\pi}{2} \right] \\ &= \frac{1}{n\pi} - \frac{2}{n^2 \pi^2} - \frac{1}{n\pi} \sin \frac{n\pi}{2} = \frac{1}{\pi^2} \left(\frac{n\pi - 2}{n^2} - \frac{\pi}{n} \sin \frac{n\pi}{2} \right) \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi x}{2} dx = \frac{1}{2} \left[\int_0^1 x \sin \frac{n\pi x}{2} dx + \int_1^2 1 \cdot \sin \frac{n\pi x}{2} dx \right] \\ &= \frac{1}{2} \left[-\frac{2}{n\pi} \left(x \cos \frac{n\pi x}{2} \Big|_0^1 - \int_0^1 \cos \frac{n\pi x}{2} dx \right) - \frac{2}{n\pi} \cos \frac{n\pi x}{2} \Big|_1^2 \right] \\ &= \frac{1}{2} \left[-\frac{2}{n\pi} \left(-\frac{2}{n\pi} \sin \frac{n\pi x}{2} \Big|_0^1 \right) - \frac{2}{n\pi} (\cos n\pi) \right] = \frac{1}{2} \left[\frac{4}{n^2 \pi^2} \sin \frac{n\pi}{2} - \frac{2}{n\pi} (-1)^n \right] \\ &= \frac{1}{\pi^2} \left[\frac{2}{n^2} \sin \frac{n\pi}{2} - \frac{\pi}{n} (-1)^n \right] \end{aligned}$$

$$f(x) = \frac{3}{8} + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \left\{ \left(\frac{n\pi - 2}{n^2} - \frac{\pi}{n} \sin \frac{n\pi}{2} \right) \cos \frac{n\pi x}{2} + \left[\frac{2}{n^2} \sin \frac{n\pi}{2} - \frac{\pi}{n} (-1)^n \right] \sin \frac{n\pi x}{2} \right\}$$