(a)Find the Fourier series of periodic function $f(x)=\begin{cases} k, & \text{if } -\pi/2<x<\pi/2 \\ 0, & \text{if } \pi/2<x<3\pi/2 \end{cases}$. (b)Show $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$. [91 清大動機 5]

[解]令 $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$$a_0=\frac{1}{\pi}\int_{-\pi/2}^{3\pi/2}f(x)dx=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}k\cdot dx=\frac{k}{\pi}\cdot x\Big|_{-\pi/2}^{\pi/2}=k$$

$$a_n=\frac{1}{\pi}\int_{-\pi/2}^{3\pi/2}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}k\cos nxdx=\frac{k}{n\pi}\cdot\sin nx\Big|_{-\pi/2}^{\pi/2}=\frac{k}{n\pi}[\sin\frac{n\pi}{2}-\sin(-\frac{n\pi}{2})]$$

$$=\frac{2k}{n\pi}\sin\frac{n\pi}{2}=\begin{cases} 0, & n\text{為偶數} \\ \frac{2k}{n\pi}, & n=1,5,9,\cdots \\ -\frac{2k}{n\pi}, & n=3,7,11,\cdots \end{cases}=\frac{2k(-1)^{n-1}}{(2n-1)\pi}$$

$$b_n=\frac{1}{\pi}\int_{-\pi/2}^{3\pi/2}f(x)\sin nxdx=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}k\sin nxdx=-\frac{k}{n\pi}\cdot\cos nx\Big|_{-\pi/2}^{\pi/2}=0$$

$$f(x)=\frac{k}{2}+\frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos(2n-1)x$$

令 $x=0$ 代入

$$\frac{f(0^-)+f(0^+)}{2}=\frac{k}{2}+\frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\Rightarrow\frac{k+k}{2}=\frac{k}{2}+\frac{2k}{\pi}(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots)$$

$$\frac{k}{2}=\frac{2k}{\pi}(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots)\Rightarrow\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$$