Find the Fourier series representation of the square wave which is given $f(x)=\begin{cases}3, & -\pi \le x < 0 \\ 5, & 0 < x < \pi\end{cases}$ and $f(x+2\pi)=f(x)$. [86 清大動機 4]

[解] $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\left(\int_{-\pi}^{0}3dx+\int_{0}^{\pi}5dx\right)=\frac{1}{\pi}\left(3x\Big|_{-\pi}^{0}+5x\Big|_{0}^{\pi}\right)=\frac{1}{\pi}(3\pi+5\pi)=8$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left(\int_{-\pi}^{0}3\cos nxdx+\int_{0}^{\pi}5\cos nxdx\right)=\frac{1}{\pi}\left(\frac{3\sin nx}{n}\Big|_{-\pi}^{0}+\frac{5\sin nx}{n}\Big|_{0}^{\pi}\right)=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\left(\int_{-\pi}^{0}3\sin nxdx+\int_{0}^{\pi}5\sin nxdx\right)=\frac{1}{\pi}\left(-\frac{3\cos nx}{n}\Big|_{-\pi}^{0}-\frac{5\cos nx}{n}\Big|_{0}^{\pi}\right)$$

$$=\frac{1}{\pi}\left(-\frac{3-3\cos n\pi}{n}-\frac{5\cos n\pi-5}{n}\right)=\frac{1}{\pi}\left(\frac{2-2\cos n\pi}{n}\right)=\frac{2[1-(-1)^n]}{n\pi}=\begin{cases}0, & n\text{為偶數}\\ \dfrac{4}{n\pi}, & n\text{為奇數}\end{cases}$$

$$=\frac{4}{(2n-1)\pi}$$

$$f(x)=4+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$$