Determine the Fourier transform of $x(t)=\begin{cases}1+\cos\pi t, & |t|\le 1\\ 0, & |t|>1\end{cases}$. [106 台大海洋丙 2]

[解]設 $x(t)=\int_{-\infty}^{\infty}c(\omega)e^{i\omega t}d\omega$

先推導$\int e^{ax}e^{ibx}dx=\int e^{(a+ib)x}dx=\dfrac{e^{(a+ib)x}}{a+ib}=\dfrac{(a-ib)e^{(a+ib)x}}{a^2+b^2}=\dfrac{e^{ax}(a-ib)(\cos bx+i\sin bx)}{a^2+b^2}$

$$=\frac{e^{ax}[(a\cos bx+b\sin bx)+i(a\sin bx-b\cos bx)]}{a^2+b^2}$$

實部$\int e^{ax}\cos bxdx=\dfrac{e^{ax}(a\cos bx+b\sin bx)}{a^2+b^2}$, 虛部$\int e^{ax}\sin bxdx=\dfrac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}$

$$c(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}x(t)e^{-i\omega t}dt=\frac{1}{2\pi}\int_{-1}^{1}(1+\cos\pi t)e^{-i\omega t}dt=\frac{1}{2\pi}\left(\int_{-1}^{1}e^{-i\omega t}dt+\int_{-1}^{1}\cos\pi t e^{-i\omega t}dt\right)$$

$$=\frac{1}{2\pi}\left[\frac{e^{-i\omega t}}{-i\omega}\Bigg|_{-1}^{1}+\frac{e^{-i\omega t}(-i\omega\cos\pi t+\pi\sin\pi t)}{(-i\omega)^2+\pi^2}\Bigg|_{-1}^{1}\right]$$

$$=\frac{1}{2\pi}\left[\frac{e^{-i\omega}-e^{i\omega}}{-i\omega}+\frac{e^{-i\omega}(i\omega)-e^{i\omega}(i\omega)}{\pi^2-\omega^2}\right]=\frac{1}{2\pi}\left[\frac{-2i\sin\omega}{-i\omega}+\frac{i\omega(-2i\sin\omega)}{\pi^2-\omega^2}\right]$$

$$=\frac{1}{\pi}\left(\frac{\sin\omega}{\omega}+\frac{\omega\sin\omega}{\pi^2-\omega^2}\right)$$

$$x(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\omega}{\omega}+\frac{\omega\sin\omega}{\pi^2-\omega^2}\right)e^{i\omega t}d\omega$$