

已知週期函數 $f(t) = \begin{cases} 0, & -\pi < t < -\frac{\pi}{2} \\ \pi, & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < t < \pi \end{cases}$, $f(t) = f(t + 2\pi)$, 試求其傅立葉級數, 並利用此結果證明

等式 $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$ 。 [104 屏科大車輛 7]

[解] $f(t)$ 為偶函數 \Rightarrow 設 $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$

$$a_0 = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \pi dt = 2t \Big|_0^{\frac{\pi}{2}} = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \pi \cos ntdt = \frac{2}{n} \cdot \sin nt \Big|_0^{\frac{\pi}{2}} = \frac{2}{n} \cdot \sin \frac{n\pi}{2} = \frac{2(-1)^{n-1}}{2n-1}$$

$$\therefore f(t) = \frac{\pi}{2} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos nt$$

$$t = 0 \text{ 代入 } \Rightarrow \frac{f(0^-) + f(0^+)}{2} = \frac{\pi}{2} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \Rightarrow \frac{\pi + \pi}{2} = \frac{\pi}{2} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} \Rightarrow \Rightarrow \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$