

I. Write down the Fourier series expansion formula of a periodic function $f(t)$ with a period 2π . II. Determine the Fourier series representation of the periodic function $f(t) = e^t$ for $-\pi < t < \pi$ and $f(t + 2\pi) = f(t)$. [104 中正機械 2(c)]

$$[\text{解}] \text{I. } f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$\text{II. } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^t dt = \frac{e^{\pi} - e^{-\pi}}{\pi} = \frac{2 \sinh \pi}{\pi}$$

$$\begin{aligned} \int e^t e^{int} dt &= \int e^{(1+in)t} dt = \frac{e^{(1+in)t}}{1+in} = \frac{(1+in)e^{(1+in)t}}{1+n^2} = \frac{e^t(1+in)(\cos nt + i \sin nt)}{1+n^2} \\ &= \frac{e^t[(\cos nt - n \sin nt) + i(n \cos nt + \sin nt)]}{1+n^2} \end{aligned}$$

$$\text{而 } \int e^t e^{int} dt = \int e^t (\cos nt + i \sin nt) dt = \int e^t \cos nt dt + i \int e^t \sin nt dt$$

$$\text{得 } \int e^t \cos nt dt = \frac{e^t(\cos nt - n \sin nt)}{1+n^2}, \int e^t \sin nt dt = \frac{e^t(n \cos nt + \sin nt)}{1+n^2}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^t \cos nt dt = \frac{1}{\pi} \cdot \frac{e^t(\cos nt - n \sin nt)}{1+n^2} \Bigg|_{-\pi}^{\pi} = \frac{e^{\pi} \cos n\pi - e^{-\pi} \cos(-n\pi)}{\pi} \\ &= \frac{(e^{\pi} - e^{-\pi})(-1)^n}{\pi} = \frac{(-1)^n \cdot 2 \sinh \pi}{\pi} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^t \sin nt dt = \frac{1}{\pi} \cdot \frac{e^t(n \cos nt + \sin nt)}{1+n^2} \Bigg|_{-\pi}^{\pi} = \frac{e^{\pi} \cdot n \cos n\pi - e^{-\pi} \cdot n \cos(-n\pi)}{\pi} \\ &= \frac{n(e^{\pi} - e^{-\pi})(-1)^n}{\pi} = \frac{(-1)^n \cdot n \cdot 2 \sinh \pi}{\pi} \end{aligned}$$

$$f(t) = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} [(-1)^n (\cos nt + n \sin nt)]$$