

If  $r(x) = x^2$ ,  $0 < x < 2\pi$ ,  $r(x) = r(x + 2\pi)$ . (a) Find  $r(x)$  in the Fourier series. (b) Evaluate  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ .

(c) Evaluate  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ . (d) Evaluate  $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$ . [104 中央機械甲乙丙能源光機電乙 2]

[解](a) 令  $r(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} x^2 dx = \frac{1}{\pi} \cdot \frac{x^3}{3} \Big|_0^{2\pi} = \frac{8\pi^2}{3}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx dx = \frac{1}{n\pi} (x^2 \sin nx \Big|_0^{2\pi} - 2 \int_0^{2\pi} x \sin nx dx) \\ &= \frac{2}{n^2 \pi} (x \cos nx \Big|_0^{2\pi} - \int_0^{2\pi} \cos nx dx) = \frac{4}{n^2} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx dx = -\frac{1}{n\pi} (x^2 \cos nx \Big|_0^{2\pi} - 2 \int_0^{2\pi} x \cos nx dx) \\ &= -\frac{4\pi}{n} + \frac{2}{n\pi} (x \sin nx \Big|_0^{2\pi} - \int_0^{2\pi} \sin nx dx) = -\frac{4\pi}{n} \end{aligned}$$

$$r(x) = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \left( \frac{1}{n^2} \cos nx - \frac{\pi}{n} \sin nx \right)$$

(b) 令  $x = 2\pi$  代入(a)得

$$\frac{r(2\pi^-) + r(2\pi^+)}{2} = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos 2n\pi \Rightarrow \frac{4\pi^2 + 0}{2} = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \dots \dots \dots \text{(i)}$$

(c) 令  $x = \pi$  代入(a)得

$$\frac{r(\pi^-) + r(\pi^+)}{2} = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi \Rightarrow \frac{\pi^2 + \pi^2}{2} = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \dots \dots \dots \text{(ii)}$$

(d) 將(i)式與(ii)式相加得

$$2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{6} + \frac{\pi^2}{12} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$