

Expand $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \leq x < \pi \end{cases}$, in a Fourier series. [102 中原生醫 4、104 中原土木 3]

$$[\text{解}] f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} (\pi - x) dx = \frac{1}{\pi} \left(\pi x - \frac{x^2}{2} \Big|_0^{\pi} \right) = \frac{\pi}{2}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{\pi} (\pi - x) \cos nx dx = \frac{1}{n\pi} [(\pi - x) \sin nx \Big|_0^{\pi} + \int_0^{\pi} \sin nx dx] \\ &= \frac{1}{n\pi} \left[0 - \frac{\cos nx}{n} \Big|_0^{\pi} \right] = -\frac{\cos n\pi - 1}{n^2 \pi} = -\frac{(-1)^n - 1}{n^2 \pi} = \frac{1 - (-1)^n}{n^2 \pi} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{\pi} (\pi - x) \sin nx dx = -\frac{1}{n\pi} [(\pi - x) \cos nx \Big|_0^{\pi} + \int_0^{\pi} \cos nx dx] \\ &= -\frac{1}{n\pi} (-\pi + 0) = \frac{1}{n} \end{aligned}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^n}{n^2 \pi} \cos nx + \frac{1}{n} \sin nx \right], \quad -\pi < x < \pi$$