Please use the Fourier integral to show that $\int_0^\infty \frac{\sin \pi\omega \sin x\omega}{1-\omega^2} d\omega = \begin{cases} \frac{\pi}{2}\sin x, & \text{if } 0 \le x \le \pi \\ 0, & \text{if } x > \pi \end{cases}$. [101 暨南電機 5]

[解]設 $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases} \Rightarrow$ 令 $f(x) = \int_0^\infty b(\omega) \sin \omega x d\omega$

$$b(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin \omega x dx = \frac{2}{\pi}\int_0^\pi \sin x \sin \omega x dx$$

$$= \frac{1}{\pi}\int_0^\pi \cos(1-\omega)x - \cos(1+\omega)x dx = \frac{1}{\pi}\cdot[\frac{\sin(1-\omega)x}{1-\omega} - \frac{\sin(1+\omega)x}{1+\omega}]\Big|_0^\pi$$

$$= \frac{1}{\pi}[\frac{\sin(1-\omega)\pi}{1-\omega} - \frac{\sin(1+\omega)\pi}{1+\omega}]$$

$$= \frac{1}{\pi}(\frac{\sin\pi\cos\omega\pi - \cos\pi\sin\omega\pi}{1-\omega} - \frac{\sin\pi\cos\omega\pi + \cos\pi\sin\omega\pi}{1+\omega}) = \frac{1}{\pi}(\frac{\sin\omega\pi}{1-\omega} - \frac{-\sin\omega\pi}{1+\omega})$$

$$= \frac{\sin\omega\pi}{\pi}(\frac{1}{1-\omega} + \frac{1}{1+\omega}) = \frac{\sin\omega\pi}{\pi}\frac{2}{1-\omega^2}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\pi}{1-\omega^2}\sin\omega x d\omega$$

當 $0 < x < \pi$ 時，$\frac{f(x^-)+f(x^+)}{2} = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\pi}{1-\omega^2}\sin\omega x d\omega$

$$\frac{\sin x + \sin x}{2} = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\pi}{1-\omega^2}\sin\omega x d\omega \Rightarrow \int_0^\infty \frac{\sin\omega\pi}{1-\omega^2}\sin\omega x d\omega = \frac{\pi}{2}\sin x$$

當 $x = 0, \pi$ 時，$\frac{f(x^-)+f(x^+)}{2} = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\pi}{1-\omega^2}\sin\omega x d\omega \Rightarrow \frac{0+0}{2} = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\pi}{1-\omega^2}\sin\omega x d\omega$

$$\int_0^\infty \frac{\sin\omega\pi}{1-\omega^2}\sin\omega x d\omega = 0$$

當 $x > \pi$ 時，$\frac{f(x^-)+f(x^+)}{2} = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\pi}{1-\omega^2}\sin\omega x d\omega \Rightarrow \frac{0+0}{2} = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\pi}{1-\omega^2}\sin\omega x d\omega$

$$\int_0^\infty \frac{\sin\omega\pi}{1-\omega^2}\sin\omega x d\omega = 0$$

因此 $\int_0^\infty \frac{\sin\omega\pi}{1-\omega^2}\sin\omega x d\omega = \begin{cases} \frac{\pi}{2}\sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$