

Use the Fourier series of the function $f(x) = \begin{cases} 0, & \text{if } 0 < x < \pi \\ 1, & \text{if } \pi < x < 2\pi \end{cases}$, find the sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$.

[101 宜蘭電機 5]

[解] 令 $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_{\pi}^{2\pi} 1 \cdot dx = \frac{1}{\pi} \cdot x \Big|_{\pi}^{2\pi} = 1$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nxdx = \frac{1}{\pi} \int_{\pi}^{2\pi} 1 \cdot \cos nxdx = \frac{1}{n\pi} \cdot \sin nx \Big|_{\pi}^{2\pi} = 0$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nxdx = \frac{1}{\pi} \int_{\pi}^{2\pi} 1 \cdot \sin nxdx = -\frac{1}{n\pi} \cdot \cos nx \Big|_{\pi}^{2\pi} = -\frac{1}{n\pi} \cdot (\cos 2n\pi - \cos n\pi) \\ &= -\frac{1}{n\pi} \cdot [1 - (-1)^n] = -\frac{2}{(2n-1)\pi} \end{aligned}$$

$$f(x) = \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x$$

令 $x = \frac{\pi}{2}$ 代入

$$\frac{f\left(\frac{\pi}{2}^{-}\right) + f\left(\frac{\pi}{2}^{+}\right)}{2} = \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi}{2} \Rightarrow \frac{0+0}{2} = \frac{1}{2} - \frac{2}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$