

Expand $f(x) = x^2$ for $0 < x < L$, in a Fourier series. [100清大動機7(c)]

$$[\text{解}] f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right)$$

$$a_0 = \frac{2}{L} \int_0^L f(x) dx = \frac{2}{L} \int_0^L x^2 dx = \frac{2}{L} \cdot \left. x^3 \right|_0^L = \frac{2}{L} \cdot \frac{L^3}{3} = \frac{2L^2}{3}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L} dx = \frac{2}{L} \int_0^L x^2 \cos \frac{2n\pi x}{L} dx \\ &= \frac{2}{L} \cdot \frac{L}{2n\pi} \left(x^2 \sin \frac{2n\pi x}{L} \right) \Big|_0^L - 2 \int_0^L x \sin \frac{2n\pi x}{L} dx \\ &= \frac{1}{n\pi} \left[2 \cdot \frac{L}{2n\pi} \left(x \cos \frac{2n\pi x}{L} \right) \Big|_0^L - \int_0^L \cos \frac{2n\pi x}{L} dx \right] = \frac{1}{n\pi} \left(\frac{L}{n\pi} \cdot L \right) = \frac{L^2}{n^2 \pi^2} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L} dx = \frac{2}{L} \int_0^L x^2 \sin \frac{2n\pi x}{L} dx \\ &= -\frac{2}{L} \cdot \frac{L}{2n\pi} \left(x^2 \cos \frac{2n\pi x}{L} \right) \Big|_0^L - 2 \int_0^L x \cos \frac{2n\pi x}{L} dx \\ &= -\frac{1}{n\pi} \left[L^2 - 2 \cdot \frac{L}{2n\pi} \left(x \sin \frac{2n\pi x}{L} \right) \Big|_0^L - \int_0^L \sin \frac{2n\pi x}{L} dx \right] \\ &= -\frac{1}{n\pi} \left[L^2 + \frac{L}{n\pi} \int_0^L \sin \frac{2n\pi x}{L} dx \right] = -\frac{1}{n\pi} \left[L^2 - \frac{L}{n\pi} \cdot \frac{L}{2n\pi} \cos \frac{2n\pi x}{L} \Big|_0^L \right] = -\frac{L^2}{n\pi} \end{aligned}$$

$$f(x) = \frac{L^2}{3} + \sum_{n=1}^{\infty} \left(\frac{L^2}{n^2 \pi^2} \cos \frac{2n\pi x}{L} - \frac{L^2}{n\pi} \sin \frac{2n\pi x}{L} \right), \quad 0 < x < L$$