

Find the Laurent expansion of the complex function  $f(z) = \frac{z+2}{z^2+4}$ . [98 成大機械 6(1)]

$$[\text{解}] f(z) = \frac{z+2}{(z+2i)(z-2i)} = \frac{A}{z+2i} + \frac{B}{z-2i} \Rightarrow z+2 = A(z-2i) + B(z+2i)$$

$$z = -2i \Rightarrow -2i + 2 = A(-4i) \Rightarrow A = \frac{-2i + 2}{-4i} = \frac{2 + 2i}{4} = \frac{1+i}{2}$$

$$z = 2i \Rightarrow 2i + 2 = B(4i) \Rightarrow B = \frac{2i + 2}{4i} = \frac{-2 + 2i}{-4} = \frac{1-i}{2}$$

$$f(z) = \frac{1+i}{2} \cdot \frac{1}{z+2i} + \frac{1-i}{2} \cdot \frac{1}{z-2i} = \frac{1+i}{2} \cdot \frac{1}{(z-2i)+4i} + \frac{1-i}{2} \cdot \frac{1}{z-2i}$$

當  $|z-2i| < 4$ ,

$$f(z) = \frac{1+i}{8i} \cdot \frac{1}{\frac{z-2i}{4i} + 1} + \frac{1-i}{2} \cdot \frac{1}{z-2i}$$

$$= \frac{1+i}{8i} \left[ 1 - \left(\frac{z-2i}{4i}\right) + \left(\frac{z-2i}{4i}\right)^2 - \left(\frac{z-2i}{4i}\right)^3 + \dots \right] + \frac{1-i}{2} \cdot \frac{1}{z-2i}$$

當  $|z-2i| > 4$ ,

$$f(z) = \frac{1+i}{2(z-2i)} \cdot \frac{1}{1 + \frac{4i}{z-2i}} + \frac{1-i}{2} \cdot \frac{1}{z-2i}$$

$$= \frac{1+i}{2(z-2i)} \left[ 1 - \left(\frac{4i}{z-2i}\right) + \left(\frac{4i}{z-2i}\right)^2 - \left(\frac{4i}{z-2i}\right)^3 + \dots \right] + \frac{1-i}{2} \cdot \frac{1}{z-2i}$$

$$= \frac{1}{z-2i} - \frac{1+i}{2} \left[ \frac{4i}{(z-2i)^2} - \frac{(4i)^2}{(z-2i)^3} + \frac{(4i)^3}{(z-2i)^4} - \frac{(4i)^4}{(z-2i)^5} + \dots \right]$$