Evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2}dz$, where C is the circle $|z|=4$; $z=x+yi$. [98 交大機械甲 5]

[解]令 $f(z)=\frac{e^z}{(z^2+\pi^2)^2}$ 有二階極點在 $z=\pm\pi i$

$$R_{\pi i}=\frac{1}{1!}\frac{d}{dz}[(z-\pi i)^2\frac{e^z}{(z^2+\pi^2)^2}]\Big|_{z=\pi i}=\frac{d}{dz}[\frac{e^z}{(z+\pi i)^2}]\Big|_{z=\pi i}=\frac{e^z[(z+\pi i)^2-2(z+\pi i)]}{(z+\pi i)^4}\Big|_{z=\pi i}$$

$$=\frac{e^{\pi i}(-4\pi^2-4\pi i)}{16\pi^4}=\frac{4\pi^2+4\pi i}{16\pi^4}$$

$$R_{-\pi i}=\frac{1}{1!}\frac{d}{dz}[(z+\pi i)^2\frac{e^z}{(z^2+\pi^2)^2}]\Big|_{z=-\pi i}=\frac{d}{dz}[\frac{e^z}{(z-\pi i)^2}]\Big|_{z=-\pi i}=\frac{e^z[(z-\pi i)^2-2(z-\pi i)]}{(z-\pi i)^4}\Big|_{z=-\pi i}$$

$$=\frac{e^{-\pi i}(-4\pi^2+4\pi i)}{16\pi^4}=\frac{4\pi^2-4\pi i}{16\pi^4}$$

$$\oint_C \frac{e^z}{(z^2+\pi^2)^2}dz=2\pi i(R_{\pi i}+R_{-\pi i})=2\pi i(\frac{4\pi^2+4\pi i}{16\pi^4}+\frac{4\pi^2-4\pi i}{16\pi^4})=\frac{i}{\pi}$$