Evaluate the real integral $\int_0^{2\pi} \frac{d\theta}{\cos\theta + \sin\theta}$. [97 台大機械 6(5)]

[解]令 $z = e^{i\theta} \Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{(e^{i\theta})^2 + 1}{2e^{i\theta}} = \frac{z^2 + 1}{2z}$, $dz = ie^{i\theta} d\theta = izd\theta \Rightarrow d\theta = \frac{dz}{iz}$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{(e^{i\theta})^2 - 1}{2ie^{i\theta}} = \frac{z^2 - 1}{2iz}$$

$$\frac{d\theta}{\cos\theta + \sin\theta} = \frac{\frac{dz}{iz}}{\frac{z^2+1}{2z} + \frac{z^2-1}{2iz}} = \frac{2dz}{i(z^2+1) + (z^2-1)} = \frac{2dz}{(i+1)z^2 + (i-1)} = \frac{2(1-i)dz}{2z^2 + 2i} = \frac{(1-i)dz}{z^2 + i}$$

令 $f(z) = \frac{1-i}{z^2 + i} \Rightarrow f(z)$ 在 $z_1 = e^{i3\pi/4}$, $z_2 = e^{i7\pi/4}$ 有單極點

$$\mathrm{Res}[f(z); z_1] = \frac{1-i}{2z}\Big|_{z=z_1} = \frac{1-i}{2e^{i3\pi/4}} = \frac{1-i}{2(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)} = \frac{1-i}{-\sqrt{2} + \sqrt{2}i} = -\frac{1}{\sqrt{2}}$$

$$\mathrm{Res}[f(z); z_2] = \frac{1-i}{2z}\Big|_{z=z_2} = \frac{1-i}{2e^{i7\pi/4}} = \frac{1-i}{2(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i)} = \frac{1-i}{\sqrt{2} - \sqrt{2}i} = \frac{1}{\sqrt{2}}$$

$$\int_0^{2\pi} \frac{d\theta}{\cos\theta + \sin\theta} = 2\pi i\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 0$$