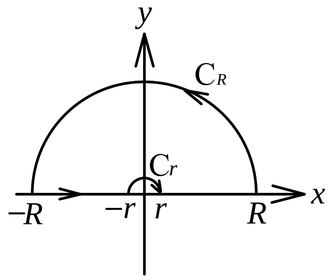
Find the integral $\int_0^\infty \frac{(\ln x)^2}{2^2+x^2}dx$. [97 台大工科海洋 6]



[解]令 $f_1(z)=\frac{\ln z}{2^2+z^2}$，$f_1(z)$在上半平面有單極點在 $z=2i$

$$\text{Res}[f_1(z);2i]=\left.\frac{\ln z}{2z}\right|_{z=2i}=\frac{\ln 2+i\frac{\pi}{2}}{4i}$$

$$\oint f_1(z)dz=2\pi i\,\text{Res}[f_1(z);2i]$$

$$\int_{C_R} f_1(z)dz+\int_{-R}^{-r}\frac{\ln x+i\pi}{2^2+x^2}dx+\int_{C_r} f_1(z)dz+\int_r^R\frac{\ln x}{2^2+x^2}dx$$

$$=\pi[\frac{\ln 2+i\frac{\pi}{2}}{2}]$$

當 $r\to 0,\ R\to\infty$時

$$0+\int_{-\infty}^0\frac{\ln x+i\pi}{2^2+x^2}dx+0+\int_0^\infty\frac{\ln x}{2^2+x^2}dx=\pi[\frac{\ln 2+i\frac{\pi}{2}}{2}]$$

$$\int_{-\infty}^\infty\frac{\ln x}{2^2+x^2}dx+i\pi\int_{-\infty}^0\frac{1}{2^2+x^2}dx=\pi[\frac{\ln 2+i\frac{\pi}{2}}{2}]\cdots\cdots\text{(i)}$$

其中

$$\int_{-\infty}^0\frac{1}{2^2+x^2}dx=\frac{1}{2}\int_{-\infty}^\infty\frac{1}{2^2+x^2}dx=\pi i\,\text{Res}[\frac{1}{2^2+z^2};2i]$$

$$=\pi i\cdot\left.\frac{1}{2z}\right|_{z=2i}=\pi i\cdot\frac{1}{4i}=\frac{\pi}{4}$$

$$\text{(i)}\Rightarrow\int_{-\infty}^\infty\frac{\ln x}{2^2+x^2}dx+i\pi\cdot\frac{\pi}{4}=\pi[\frac{\ln 2+i\frac{\pi}{2}}{2}]$$

$$\int_{-\infty}^\infty\frac{\ln x}{2^2+x^2}dx+i\pi\cdot\frac{\pi}{4}=\frac{\pi\ln 2}{2}+i\pi\cdot\frac{\pi}{4}$$

得$\int_{-\infty}^\infty\frac{\ln x}{2^2+x^2}dx=\frac{\pi\ln 2}{2}\Rightarrow\int_{-\infty}^0\frac{\ln x}{2^2+x^2}dx=\frac{\pi\ln 2}{4}$

令 $f_2(z)=\dfrac{(\ln z)^2}{2^2+z^2}$，$f_2(z)$在上半平面有單極點在 $z=2i$

$$\mathrm{Res}[f_2(z);2i]=\left.\frac{(\ln z)^2}{2z}\right|_{z=2i}=\frac{(\ln 2+i\frac{\pi}{2})^2}{4i}=\frac{(\ln 2)^2+i\pi\ln 2-\pi^2/4}{4i}$$

$$\oint f_2(z)dz=2\pi i\,\mathrm{Res}[f_2(z);2i]$$

$$\int_{C_R} f_2(z)dz+\int_{-R}^{-r}\frac{(\ln x+i\pi)^2}{2^2+x^2}dx+\int_{C_r} f_2(z)dz+\int_r^R\frac{(\ln x)^2}{2^2+x^2}dx=\pi[\frac{(\ln 2)^2+i\pi\ln 2-\pi^2/4}{2}]$$

當 $r\to 0,\ R\to\infty$ 時

$$0+\int_{-\infty}^{0}\frac{(\ln x)^2+i2\pi\ln x-\pi^2}{2^2+x^2}dx+0+\int_0^{\infty}\frac{(\ln x)^2}{2^2+x^2}dx=\pi[\frac{(\ln 2)^2+i\pi\ln 2-\pi^2/4}{2}]$$

$$\int_{-\infty}^{\infty}\frac{(\ln x)^2}{2^2+x^2}dx+i2\pi\int_{-\infty}^{0}\frac{\ln x}{2^2+x^2}dx-\pi^2\int_{-\infty}^{0}\frac{1}{2^2+x^2}dx=\pi[\frac{(\ln 2)^2+i\pi\ln 2-\pi^2/4}{2}]$$

$$\int_{-\infty}^{\infty}\frac{(\ln x)^2}{2^2+x^2}dx+i2\pi\cdot\frac{\pi\ln 2}{4}-\pi^2\cdot\frac{\pi}{4}=\frac{\pi(\ln 2)^2}{2}+\frac{i\pi^2\ln 2}{2}-\frac{\pi^3}{8}$$

$$\int_{-\infty}^{\infty}\frac{(\ln x)^2}{2^2+x^2}dx=\frac{\pi(\ln 2)^2}{2}+\frac{\pi^3}{8}\Rightarrow\int_0^{\infty}\frac{(\ln x)^2}{2^2+x^2}dx=\frac{\pi(\ln 2)^2}{4}+\frac{\pi^3}{16}$$