Evaluate $\int_0^{\pi} \frac{\cos 2\theta}{2-\cos\theta} d\theta$. [96 中央機械能源三 3]

[解]令 $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta = izd\theta \Rightarrow d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+1/z}{2} = \frac{z^2+1}{2z}$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2(\frac{z^2+1}{2z})^2 - 1 = \frac{z^4+2z^2+1}{2z^2} - 1 = \frac{z^4+1}{2z^2}$$

$$\frac{\cos 2\theta}{2-\cos\theta} d\theta = \frac{\frac{z^4+1}{2z^2}}{2-\frac{z^2+1}{2z}} \frac{dz}{iz} = \frac{z^4+1}{4z^2 - z(z^2+1)} \frac{dz}{iz} = \frac{i(z^4+1)}{z^4-4z^3+z^2} dz = \frac{i(z^4+1)}{z^2(z^2-4z+1)} dz$$

單位圓內有單極點 $z = 2-\sqrt{3}$ 及二階極點 $z = 0$

$$R_{2-\sqrt{3}} = \left.\frac{i(z^4+1)}{4z^3-12z^2+2z}\right|_{z=2-\sqrt{3}} = \frac{i(98-56\sqrt{3})}{24-14\sqrt{3}} = \frac{i(98-56\sqrt{3})}{24-14\sqrt{3}} = \frac{14i(7-4\sqrt{3})}{2\sqrt{3}(4\sqrt{3}-7)} = -\frac{7\sqrt{3}}{3} i$$

$$R_0 = \left.\frac{d}{dz}[z^2 \cdot \frac{i(z^4+1)}{z^2(z^2-4z+1)}]\right|_{z=0} = \left.\frac{i\cdot 4z^3\cdot(z^2-4z+1) - i(z^4+1)\cdot(2z-4)}{(z^2-4z+1)^2}]\right|_{z=0} = 4i$$

$$\int_0^{\pi} \frac{\cos 2\theta}{2-\cos\theta} d\theta = \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta}{2-\cos\theta} d\theta = \pi i(R_{2-\sqrt{3}} + R_0) = \pi i(-\frac{7\sqrt{3}}{3} i + 4i) = \frac{7\sqrt{3}-12}{3}\pi$$