Find $\int_0^{\infty}\frac{dx}{x^6+1}$. [94 中央機械 9(c)]

[解]令 $f(z)=\frac{z^2+1}{z^6+1}$

$f(z)$在上半平面有單極點 $z=e^{i\frac{\pi}{6}},e^{i\frac{\pi}{2}},e^{i\frac{5\pi}{6}}$

$$R_{e^{i\frac{\pi}{6}}}=\left.\frac{1}{6z^5}\right|_{z=e^{i\frac{\pi}{6}}}=\frac{1}{6e^{i\frac{5\pi}{6}}}=\frac{1}{6(-\frac{\sqrt{3}}{2}+\frac{1}{2}i)}=\frac{1}{-3\sqrt{3}+3i}=\frac{-3\sqrt{3}-3i}{(-3\sqrt{3})^2+3^2}$$

$$=\frac{-3\sqrt{3}-3i}{36}=\frac{-\sqrt{3}-i}{12}$$

$$R_{e^{i\frac{\pi}{2}}}=\left.\frac{1}{6z^5}\right|_{z=e^{i\frac{\pi}{2}}}=\frac{1}{6e^{i\frac{5\pi}{2}}}=\frac{1}{6(0+i)}=-\frac{i}{6}$$

$$R_{e^{i\frac{5\pi}{6}}}=\left.\frac{1}{6z^5}\right|_{z=e^{i\frac{5\pi}{6}}}=\frac{1}{6e^{i\frac{25\pi}{6}}}=\frac{1}{6(\frac{\sqrt{3}}{2}+\frac{1}{2}i)}=\frac{1}{3\sqrt{3}+3i}=\frac{3\sqrt{3}-3i}{(3\sqrt{3})^2+3^2}$$

$$=\frac{3\sqrt{3}-3i}{36}=\frac{\sqrt{3}-i}{12}$$

$$\int_0^{\infty}\frac{1}{x^6+1}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{x^6+1}dx=\pi i(R_{e^{i\frac{\pi}{6}}}+R_{e^{i\frac{\pi}{2}}}+R_{e^{i\frac{5\pi}{6}}})=\pi i(-\frac{i}{3})=\frac{\pi}{3}$$