

Evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2}$. [91 成大造船 8(2)]

[解] 令 $f(z) = \frac{1}{(z^2+1)(z^2+4)^2} = \frac{1}{z^6+9z^4+24z^2+16}$

$f(z)$ 在上半平面有單極點 $z=i$ ，二階極點在 $z=2i$

$$R_i = \left. \frac{1}{6z^5+36z^3+48z} \right|_{z=i} = \frac{1}{6i-36i+48i} = -\frac{i}{18}$$

$$R_{2i} = \left. \frac{1}{1!} \frac{d}{dz} \left[(z-2i)^2 \frac{1}{(z^2+1)(z^2+4)^2} \right] \right|_{z=2i} = \left. \frac{d}{dz} \left[\frac{1}{(z^2+1)(z+2i)^2} \right] \right|_{z=2i}$$
$$= \left. \frac{-[2z(z+2i)^2 + (z^2+1) \cdot 2(z+2i)]}{(z^2+1)^2(z+2i)^4} \right|_{z=2i} = \frac{-(-64i-24i)}{9 \cdot 256} = \frac{11i}{288}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2} = 2\pi i(R_i + R_{2i}) = 2\pi i \left(-\frac{i}{18} + \frac{11i}{288} \right) = \frac{5\pi}{144}$$