Evaluate the integral $\int_0^{2\pi}\frac{1+\sin\theta}{3+\cos\theta}d\theta$. [91 成大造船 8(1)]

[解]令$z=e^{i\theta}\Rightarrow dz=ie^{i\theta}d\theta=izd\theta\Rightarrow d\theta=\frac{dz}{iz}$, $\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{z+1/z}{2}=\frac{z^2+1}{2z}$

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}=\frac{z-1/z}{2i}=\frac{z^2-1}{2iz}$$

$$\frac{1+\sin\theta}{3+\cos\theta}d\theta=\frac{1+\frac{z^2-1}{2iz}}{3+\frac{z^2+1}{2z}}\frac{dz}{iz}=\frac{2iz+(z^2-1)}{6iz+i(z^2+1)}\frac{dz}{iz}=\frac{z^2+2iz-1}{-z^3-6z^2-z}dz=\frac{z^2+2iz-1}{-z(z^2+6z+1)}dz$$

單位圓內有單極點$z=0,-3+2\sqrt{2}$

$$R_0=\left.\frac{z^2+2iz-1}{-3z^2-12z-1}\right|_{z=0}=1$$

$$R_{-3+2\sqrt{2}}=\left.\frac{z^2+2iz-1}{-3z^2-12z-1}\right|_{z=-3+2\sqrt{2}}=\frac{16-12\sqrt{2}+(-6+4\sqrt{2})i}{-16+12\sqrt{2}}=-1-\frac{\sqrt{2}}{4}i$$

$\int_0^{2\pi}\frac{1+\sin\theta}{3+\cos\theta}d\theta=\oint_C\frac{z^2+2iz-1}{-z(z^2+6z+1)}dz$，其中C為圓$|z|=1$

$$\therefore\int_0^{2\pi}\frac{1+\sin\theta}{3+\cos\theta}d\theta=2\pi i(R_0+R_{-3+2\sqrt{2}})=2\pi i(-\frac{\sqrt{2}}{4}i)=\frac{\sqrt{2}\pi}{2}$$