Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta} d\theta$. [91 中央機械 2(c)]

[解]令 $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta = iz d\theta \Rightarrow d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+1/z}{2} = \frac{z^2+1}{2z}$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2(\frac{z^2+1}{2z})^2 - 1 = \frac{z^4+2z^2+1}{2z^2} - 1 = \frac{z^4+1}{2z^2}$$

$$\frac{\cos 2\theta}{5-4\cos\theta} d\theta = \frac{\frac{z^4+1}{2z^2}}{5-4\frac{z^2+1}{2z}} \frac{dz}{iz} = \frac{z^4+1}{10z^2-4z(z^2+1)} \frac{dz}{iz} = \frac{i(z^4+1)}{4z^4-10z^3+4z^2} dz$$

$$= \frac{i(z^4+1)}{2z^2(2z-1)(z-2)} dz$$

單位圓內有單極點 $z = \frac{1}{2}$ 及二階極點 $z = 0$

$$R_{\frac{1}{2}} = \left.\frac{i(z^4+1)}{16z^3-30z^2+8z}\right|_{z=\frac{1}{2}} = \frac{i(17/16)}{2-15/2+4} = \frac{i(17/16)}{-3/2} = -\frac{17}{24} i$$

$$R_0 = \frac{d}{dz}\left[z^2 \cdot \frac{i(z^4+1)}{4z^4-10z^3+4z^2}\right]\Bigg|_{z=0} = \frac{i \cdot 4z^3 \cdot (4z^2-10z+4) - i(z^4+1)\cdot(8z-10)}{(4z^2-10z+4)^2}\Bigg]\Bigg|_{z=0}$$

$$= \frac{10i}{16} = \frac{5}{8} i$$

$$\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta} d\theta = 2\pi i (R_{\frac{1}{2}} + R_0) = \pi i(-\frac{17}{24} i + \frac{5}{8} i) = \frac{\pi}{12}$$