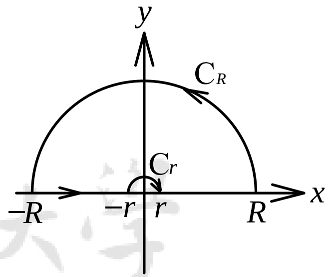
Evaluate $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx$ using complex contour integral. [90 成大機械 3(c)]

[解] $\frac{\sin^2 x}{x^2} = \frac{1-\cos 2x}{2x^2}$，令 $f(z) = \frac{1-e^{i2z}}{2z^2}$



$f(z)$ 在 $z=0$ 有二階極點

$$R_0 = \frac{1}{1!}\frac{d}{dz}[z^2 \cdot f(z)]\Big|_{z=0} = \frac{-2ie^{-2z}}{2}\Big|_{z=0} = -i$$

極點恰在實數軸上，使用避點圍線

$$\int_{-R}^{-r} \frac{1-e^{i2x}}{2x^2} dx + \int_{C_r} \frac{1-e^{i2z}}{2z^2} dz + \int_r^R \frac{1-e^{i2x}}{2x^2} dx + \int_{C_R} \frac{1-e^{i2z}}{2z^2} dz = 0$$

$$\lim_{\substack{r\to 0\\ R\to\infty}} \int_{-R}^{-r} \frac{1-e^{i2x}}{2x^2} dx - \pi i \cdot R_0 + \lim_{\substack{r\to 0\\ R\to\infty}} \int_r^R \frac{1-e^{i2x}}{2x^2} dx + 0 = 0$$

$$\int_{-\infty}^{0} \frac{1-e^{i2x}}{2x^2} dx - \pi i \cdot (-i) + \int_0^{\infty} \frac{1-e^{i2x}}{2x^2} dx + 0 = 0,$$

$$\int_{-\infty}^{\infty} \frac{1-e^{i2x}}{2x^2} dx = \pi \Rightarrow \int_{-\infty}^{\infty} \frac{(1-\cos 2x) - i\sin 2x}{2x^2} dx = \pi$$

$$\int_{-\infty}^{\infty} \frac{1-\cos 2x}{2x^2} dx = \pi, \quad \int_{-\infty}^{\infty} \frac{\sin 2x}{2x^2} dx = 0$$

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \int_0^{\infty} \frac{1-\cos 2x}{2x^2} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos 2x}{2x^2} dx = \frac{\pi}{2}$$