

Evaluate the following integration $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3}$. [89 台科大機械 4]

[解] 令 $f(z) = \frac{1}{(1+z^2)^3} = \frac{1}{[(z+i)(z-i)]^3} \Rightarrow f(z)$ 在上半平面有 3 階極點 $z = i$

$$R_i = \frac{1}{2!} \frac{d^2}{dz^2} \left[(z-i)^3 \cdot \frac{1}{[(z+i)(z-i)]^3} \right] \Bigg|_{z=i} = \frac{1}{2} \frac{d^2}{dz^2} \left[\frac{1}{(z+i)^3} \right] \Bigg|_{z=i} = \frac{1}{2} \frac{d}{dz} \left[\frac{-3(z+i)^2}{(z+i)^6} \right] \Bigg|_{z=i}$$
$$= -\frac{3}{2} \frac{d}{dz} \left[\frac{1}{(z+i)^4} \right] \Bigg|_{z=i} = -\frac{3}{2} \cdot \frac{-4(z+i)^3}{(z+i)^8} \Bigg|_{z=i} = 6 \cdot \frac{1}{(z+i)^5} \Bigg|_{z=i} = \frac{6}{32i} = -\frac{3i}{16}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3} = 2\pi i \cdot R_i = 2\pi i \left(-\frac{3i}{16} \right) = \frac{3\pi}{8}$$