

Evaluate $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$. [88 清大動機 5(b)]

[解] 令 $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta = izd\theta \Rightarrow d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z+1/z}{2} = \frac{z^2+1}{2z}$

$$\frac{d\theta}{5+3\cos\theta} = \frac{\frac{dz}{iz}}{5+3\frac{z^2+1}{2z}} = \frac{-2idz}{3z^2+10z+3} = \frac{-2idz}{(3z+1)(z+3)}$$

$\frac{-2i}{(3z+1)(z+3)}$ 在單位圓 C 內有單極點 $z = -\frac{1}{3} \Rightarrow R_{-\frac{1}{3}} = \left. \frac{-2i}{6z+10} \right|_{z=-1/3} = -\frac{i}{4}$

$$\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \oint_C \frac{-2idz}{(3z+1)(z+3)} = 2\pi i \cdot R_{-\frac{1}{3}} = 2\pi i \left(-\frac{i}{4}\right) = \frac{\pi}{2}$$