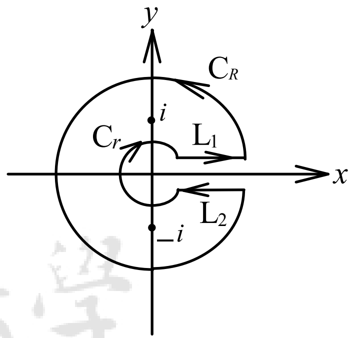
Evaluate $\int_0^\infty \frac{x^{1/3}}{x(x^2+1)}dx$. [86 台科大機械 5]



[解]令 $f(z)=\frac{z^{1/3}}{z(z^2+1)}$, $f(z)$有單極點在 $z=0, i, -i$

而 $z^{1/3}=|z|^{1/3}e^{i\theta/3}$

在 L_1 上，$f(z)\to\frac{x^{1/3}}{x(x^2+1)}$　　　在 L_2 上，$f(z)\to\frac{x^{1/3}e^{i2\pi/3}}{x(x^2+1)}$

$R_i=\frac{i^{1/3}}{3i^2+1}=\frac{e^{i\pi/6}}{-2}$　　　$R_{-i}=\frac{(-i)^{1/3}}{3(-i)^2+1}=\frac{e^{i\pi/2}}{-2}$

$$\oint f(z)dz=2\pi i(R_i+R_{-i})$$

$$\int_{C_R} f(z)dz+\int_R^r \frac{x^{1/3}e^{i2\pi/3}}{x(x^2+1)}dx+\int_{C_r} f(z)dz+\int_r^R \frac{x^{1/3}}{x(x^2+1)}dx=\frac{e^{i\pi/6}+e^{i\pi/2}}{-2}$$

當 $r\to 0, R\to\infty$ 時

$$0+e^{i2\pi/3}\int_\infty^0 \frac{x^{1/3}}{x(x^2+1)}dx+0+\int_0^\infty \frac{x^{1/3}}{x(x^2+1)}dx=\frac{e^{i\pi/6}+e^{i\pi/2}}{-2}$$

$$(1-e^{i2\pi/3})\int_0^\infty \frac{x^{1/3}}{x(x^2+1)}dx=\frac{e^{i\pi/6}+e^{i\pi/2}}{-2}$$

$$\frac{3-\sqrt{3}i}{2}\int_0^\infty \frac{x^{1/3}}{x(x^2+1)}dx=\frac{\pi(3-\sqrt{3}i)}{2}\Rightarrow\int_0^\infty \frac{x^{1/3}}{x(x^2+1)}dx=\pi$$