Apply the residue theorem to evaluate $\int_0^{2\pi} \frac{\cos 2\theta d\theta}{1-2p\cos\theta+p^2}$, where $p>1$. [85 台科大機械 5]

[解]令 $z=e^{i\theta} \Rightarrow dz=ie^{i\theta}d\theta=izd\theta$, $\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{z+1/z}{2}=\frac{z^2+1}{2z}$

$$\cos 2\theta=2\cos^2\theta-1=2\left(\frac{z^2+1}{2z}\right)^2-1=\frac{z^4+2z^2+1}{2z^2}-1=\frac{z^4+1}{2z^2}$$

$$\frac{\cos 2\theta d\theta}{1-2p\cos\theta+p^2}=\frac{\frac{z^4+1}{2z^2}\frac{dz}{iz}}{1-2p\frac{z^2+1}{2z}+p^2}=\frac{-i(z^4+1)dz}{2z^3-2pz^2(z^2+1)+2p^2z^3}$$

$$=\frac{-i(z^4+1)dz}{2z^3(1+p^2)-2pz^2(z^2+1)}=\frac{i(z^4+1)dz}{2z^2[pz^2-(1+p^2)z+p]}=\frac{i(z^4+1)dz}{2z^2(pz-1)(z-p)}$$

有三個極點在 $z=0,\frac{1}{p},p$ (0為二階, p 在單位圓外)

$$R_0=\frac{1}{1!}\frac{d}{dz}\left[z^2\cdot\frac{i(z^4+1)}{2z^2(pz-1)(z-p)}\right]\Bigg|_{z=0}=\frac{1}{1!}\frac{d}{dz}\left[\frac{i(z^4+1)}{2(pz-1)(z-p)}\right]\Bigg|_{z=0}$$

$$=\frac{1}{1!}\frac{d}{dz}\left[\frac{i(z^4+1)}{2pz^2-2(1+p^2)z+2p}\right]\Bigg|_{z=0}$$

$$=\frac{i4z^3[2(pz-1)(z-p)]-i(z^4+1)[4pz-2(1+p^2)]}{[2(pz-1)(z-p)]^2}\Bigg|_{z=0}=\frac{0-i[-2(1+p^2)]}{[2(-1)(-p)]^2}=\frac{i(1+p^2)}{2p^2}$$

$$R_{\frac{1}{p}}=\frac{i(z^4+1)}{8pz^3-6z^2(1+p^2)+4pz}\Bigg|_{z=\frac{1}{p}}=\frac{i(\frac{1}{p^4}+1)}{\frac{8}{p^2}-\frac{6}{p^2}(1+p^2)+4}=\frac{i(1+p^4)}{8p^2-6p^2(1+p^2)+4p^4}$$

$$=\frac{i(1+p^4)}{2p^2(1-p^2)}$$

$$R_0+R_{\frac{1}{p}}=\frac{i(1+p^2)}{2p^2}+\frac{i(1+p^4)}{2p^2(1-p^2)}=\frac{i(1+p^2)(1-p^2)+i(1+p^4)}{2p^2(1-p^2)}=\frac{2i}{2p^2(1-p^2)}$$

$$\int_0^{2\pi}\frac{\cos 2\theta d\theta}{1-2p\cos\theta+p^2}=2\pi i\left[\frac{2i}{2p^2(1-p^2)}\right]=\frac{2\pi}{p^2(p^2-1)}$$