Let $F(z)=\dfrac{1}{(z^2-4)(z-3)^2}$, where z is a complex variable. (a)Identify poles and the order of poles of $F(z)$. (b)Determine the residue of each pole. (c)Determine the inverse Laplace transform of $F(z)$. Please draw the integral contour for calculating the inverse Laplace transform. [106 台大機械 7]

[解] $F(z)=\dfrac{1}{(z+2)(z-2)(z-3)^2}=\dfrac{1}{z^4-6z^3+5z^2+24z-36}$

(a) $F(z)$有極點$-2, 2, 3$，其中$-2, 2$的階數為1，3的階數為2

(b) $\text{Res}[F(z);-2]=\left.\dfrac{1}{4z^3-18z^2+10z+24}\right|_{z=-2}=-\dfrac{1}{100}$

$\text{Res}[F(z);2]=\left.\dfrac{1}{4z^3-18z^2+10z+24}\right|_{z=2}=\dfrac{1}{4}$

$\text{Res}[F(z);3]=\left.\dfrac{d}{dz}\left(\dfrac{1}{z^2-4}\right)\right|_{z=3}=\left.\dfrac{-2z}{(z^2-4)^2}\right|_{z=3}=\dfrac{-6}{25}$

(c) $\text{Res}[e^{zt}F(z);-2]=\left.\dfrac{e^{zt}}{4z^3-18z^2+10z+24}\right|_{z=-2}=-\dfrac{e^{-2t}}{100}$

$\text{Res}[e^{zt}F(z);2]=\left.\dfrac{e^{zt}}{4z^3-18z^2+10z+24}\right|_{z=2}=\dfrac{e^{2t}}{4}$

$\text{Res}[e^{zt}F(z);3]=\left.\dfrac{d}{dz}\left(\dfrac{e^{zt}}{z^2-4}\right)\right|_{z=3}=\left.\dfrac{te^{zt}(z^2-4)-2ze^{zt}}{(z^2-4)^2}\right|_{z=3}=\dfrac{te^{3t}}{5}-\dfrac{6e^{3t}}{25}$

○$^{-1}[F(z)]=\dfrac{1}{2\pi i}\cdot 2\pi i\,\{\text{Res}[e^{zt}F(z);-2]+\text{Res}[e^{zt}F(z);2]+\text{Res}[e^{zt}F(z);3]\}$

$=-\dfrac{e^{-2t}}{100}+\dfrac{e^{2t}}{4}+\dfrac{te^{3t}}{5}-\dfrac{6e^{3t}}{25}$