Evaluate $\int_{-\infty}^{\infty} \frac{\cos mx}{1+x^2} dx$. [105 成大機械 5]

[解]令 $f(z)=\frac{e^{imz}}{1+z^2}$，則$\int_{-\infty}^{\infty} \frac{\cos mxdx}{1+x^2}$為$\int_{-\infty}^{\infty} f(z)dz$的實部

(1)當$m>0$時，上半平面只有極點i

$$\text{Res}[f(z);i]=\left.\frac{e^{imz}}{2z}\right|_{z=i}=\frac{e^{i(mi)}}{2i}=-\frac{e^{-m}}{2}i$$

$$\int_{-\infty}^{\infty} \frac{\cos mxdx}{1+x^2}=\text{Re}[2\pi i\cdot(-\frac{e^{-m}}{2}i)]=\pi e^{-m}$$

(2)當$m=0$時，$f(z)=\frac{1}{1+z^2}$上半平面只有極點i

$$\text{Res}[f(z);i]=\left.\frac{1}{2z}\right|_{z=i}=\frac{1}{2i}=-\frac{1}{2}i$$

$$\int_{-\infty}^{\infty} \frac{\cos mxdx}{1+x^2}=\text{Re}[2\pi i\cdot(-\frac{1}{2}i)]=\pi$$

(3)當$m<0$時，上半平面只有極點$-i$

$$\text{Res}[f(z);i-]=\left.\frac{e^{imz}}{2z}\right|_{z=-i}=\frac{e^{i(-mi)}}{-2i}=\frac{e^{m}}{2}i$$

$$\int_{-\infty}^{\infty} \frac{\cos mxdx}{1+x^2}=\text{Re}[-2\pi i\cdot(\frac{e^{m}}{2}i)]=\pi e^{m}$$

$$\int_{-\infty}^{\infty} \frac{\cos xdx}{1+x^2}=\pi e^{-|m|}$$