The inverse Laplace transform can be written as $f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)e^{st}ds$, where the path of integral with respect to $s$ is a vertical line parallel to the imaginary axis, and is on the right of all singularities of $F(s)$ in the complex plane. Now, by use of the residue theorem, calculate the inverse Laplace transform of $\frac{s^3}{s^4-a^4}$. [104 成大機械 4]

[解]令$F(z)=\frac{z^3}{z^4-a^4}$，$F(z)$有單極點$a, ae^{i\pi/2}, ae^{i\pi}, ae^{i3\pi/2}$

$$\mathrm{Res}[e^{zt}F(z);a]=\left.\frac{e^{zt}z^3}{4z^3}\right|_{z=a}=\frac{e^{at}}{4}$$

$$\mathrm{Res}[e^{zt}F(z);ae^{i\pi/2}]=\left.\frac{e^{zt}z^3}{4z^3}\right|_{z=ae^{i\pi/2}}=\frac{e^{ae^{i\pi/2}t}}{4}=\frac{e^{iat}}{4}$$

$$\mathrm{Res}[e^{zt}F(z);ae^{i\pi}]=\left.\frac{e^{zt}z^3}{4z^3}\right|_{z=ae^{i\pi}}=\frac{e^{ae^{i\pi}t}}{4}=\frac{e^{-at}}{4}$$

$$\mathrm{Res}[e^{zt}F(z);ae^{i3\pi/2}]=\left.\frac{e^{zt}z^3}{4z^3}\right|_{z=ae^{i3\pi/2}}=\frac{e^{ae^{i3\pi/2}t}}{4}=\frac{e^{-iat}}{4}$$

$$f(t)=\frac{1}{2\pi i}\cdot 2\pi i\left[\frac{e^{at}}{4}+\frac{e^{iat}}{4}+\frac{e^{-at}}{4}+\frac{e^{-iat}}{4}\right]$$

$$=\frac{1}{4}[(e^{at}+e^{-at})+(e^{iat}+e^{-iat})]=\frac{1}{2}(\cosh at+\cos at)$$