Evaluate the value of the integral $\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4} dx \, (a>0)$. [104 中央電機五]

[解]令 $f(z)=\frac{z^3 e^{iaz}}{z^4+4}$，則$\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4} dx$為$\int_{-\infty}^{\infty} f(z)dz$的虛部

$f(z)$有單極點$\sqrt{2}e^{i\frac{\pi}{4}}=1+i, \sqrt{2}e^{i\frac{3\pi}{4}}=-1+i$在上半平面

$$R_{1+i}=\left.\frac{z^3 e^{iaz}}{4z^3}\right|_{z=1+i}=\frac{e^{ia(1+i)}}{4}=\frac{e^{-a}\cdot e^{ia}}{4}$$

$$R_{-1+i}=\left.\frac{z^3 e^{iaz}}{4z^3}\right|_{z=-1+i}=\frac{e^{ia(-1+i)}}{4}=\frac{e^{-a}\cdot e^{-ia}}{4}$$

$$\int_{-\infty}^{\infty} f(z)dz=2\pi i\cdot(R_{1+i}+R_{-1+i})=2\pi i\cdot(\frac{e^{-a}\cdot e^{ia}}{4}+\frac{e^{-a}\cdot e^{-ia}}{4})=i\frac{\pi e^{-a}}{2}(e^{ia}+e^{-ia})$$

$$=i\pi e^{-a}\cos a$$

$$\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4} dx=\text{Im}[\int_{-\infty}^{\infty} f(z)dz]=\pi e^{-a}\cos a$$