請計算積分 $\int_0^\infty \frac{x^2+1}{x^6+1}dx$. [102 海洋光電 9]

[解]令 $f(z)=\frac{z^2+1}{z^6+1}$

$f(z)$ 在上半平面有單極點 $z=e^{i\frac{\pi}{6}}, e^{i\frac{\pi}{2}}, e^{i\frac{5\pi}{6}}$

$$R_{e^{i\frac{\pi}{6}}}=\left.\frac{z^2+1}{6z^5}\right|_{z=e^{i\frac{\pi}{6}}}=\frac{e^{i\frac{\pi}{3}}+1}{6e^{i\frac{5\pi}{6}}}=\frac{(\frac{1}{2}+\frac{\sqrt{3}}{2}i)+1}{6(-\frac{\sqrt{3}}{2}+\frac{1}{2}i)}=\frac{\frac{3}{2}+\frac{\sqrt{3}}{2}i}{-3\sqrt{3}+3i}=\frac{3+\sqrt{3}i}{-6\sqrt{3}+6i}=\frac{(3+\sqrt{3}i)(-6\sqrt{3}-6i)}{(-6\sqrt{3})^2+6^2}$$

$$=\frac{-12\sqrt{3}-36i}{144}=\frac{-\sqrt{3}-3i}{12}$$

$$R_{e^{i\frac{\pi}{2}}}=\left.\frac{z^2+1}{6z^5}\right|_{z=e^{i\frac{\pi}{2}}}=\frac{e^{i\pi}+1}{6e^{i\frac{5\pi}{2}}}=\frac{(-1)+1}{6(0+i)}=0$$

$$R_{e^{i\frac{5\pi}{6}}}=\left.\frac{z^2+1}{6z^5}\right|_{z=e^{i\frac{5\pi}{6}}}=\frac{e^{i\frac{5\pi}{3}}+1}{6e^{i\frac{25\pi}{6}}}=\frac{(\frac{1}{2}-\frac{\sqrt{3}}{2}i)+1}{6(\frac{\sqrt{3}}{2}+\frac{1}{2}i)}=\frac{\frac{3}{2}-\frac{\sqrt{3}}{2}i}{3\sqrt{3}+3i}=\frac{3-\sqrt{3}i}{6\sqrt{3}+6i}=\frac{(3-\sqrt{3}i)(6\sqrt{3}-6i)}{(6\sqrt{3})^2+6^2}$$

$$=\frac{12\sqrt{3}-36i}{144}=\frac{\sqrt{3}-3i}{12}$$

$$\int_0^\infty \frac{x^2+1}{x^6+1}dx=\frac{1}{2}\int_{-\infty}^\infty \frac{x^2+1}{x^6+1}dx=\pi i(R_{e^{i\frac{\pi}{6}}}+R_{e^{i\frac{\pi}{2}}}+R_{e^{i\frac{5\pi}{6}}})=\pi i(-\frac{i}{2})=\frac{\pi}{2}$$