Determine the integral $\int_0^{\infty} \frac{x \sin x}{(x^2+1)(x^2+4)} dx$. [100 中央機械能源光機電生醫 10]

[解]令 $f(z) = \frac{ze^{iz}}{(z^2+1)(z^2+4)}$，則$\int_{-\infty}^{\infty} \frac{x \sin x dx}{(x^2+1)(x^2+4)}$為$\int_{-\infty}^{\infty} f(z)dz$的虛部

$$R_i = \left. \frac{ze^{iz}}{2z(z^2+4)+(z^2+1)\cdot 2z} \right|_{z=i} = \frac{ie^{i(i)}}{2i(-1+4)+0} = \frac{e^{-1}}{6}$$

$$R_{2i} = \left. \frac{ze^{iz}}{2z(z^2+4)+(z^2+1)\cdot 2z} \right|_{z=2i} = \frac{2ie^{i(2i)}}{0+(-4+1)\cdot 4i} = -\frac{e^{-2}}{6}$$

$$\int_0^{\infty} \frac{x \sin x dx}{(x^2+1)(x^2+4)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin x dx}{(x^2+1)(x^2+4)} = \text{Im}[2\pi i \cdot (\frac{e^{-1}}{6} - \frac{e^{-2}}{6})] = \frac{\pi}{3}\cdot(e^{-1}-e^{-2})$$