$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ \text{B.C.}:\frac{\partial u}{\partial x}(x=0)=0,\ \frac{\partial u}{\partial x}(x=a)=0,\ \text{for } 0<y<b$$

$$u(x,0)=0,\ u(x,b)=1,\ \text{for } 0<x<a$$

(1)Let $u(x,y)=X(x)Y(y)$, then show the new boundary conditions associated with X and Y.

(2)Solve u. [90雲科大機械4]

[解]令$u=XY$，代入方程式$\Rightarrow X''Y+XY''=0\Rightarrow \frac{X''}{X}=-\frac{Y''}{Y}=-\lambda^2$

得兩個方程式$X''+\lambda^2 X=0, Y''-\lambda^2 Y=0$

由邊界條件$\frac{\partial u}{\partial x}(0,y)=0, \frac{\partial u}{\partial x}(a,y)=0\Rightarrow X'(0)=0, X'(a)=0$

由X的方程式知$X=C_1\cos\lambda x+C_2\sin\lambda x$

$$\begin{cases}X'(0)=0\Rightarrow C_2\lambda=0\\ X'(a)=0\Rightarrow -C_1\lambda\sin\lambda a+C_2\lambda\cos\lambda a=0\end{cases}\Rightarrow C_2=0, \sin\lambda a=0\Rightarrow \lambda a=n\pi$$

知特徵值$\lambda_n=\frac{n\pi}{a}\Rightarrow X_n=\cos\lambda_n x$

由Y的方程式知$Y=D_n e^{\lambda_n y}+E_n e^{-\lambda_n y}$

由邊界條件$u(x,0)=0\Rightarrow Y(0)=0\Rightarrow D_n+E_n=0\Rightarrow E_n=-D_n$

$Y=2D_n\sinh(\lambda_n y)=F_n\sinh(\lambda_n y)$

$u(x,y)=\frac{F_0}{2}\cdot\frac{y}{b}+\sum_{n=1}^{\infty}F_n\cos\lambda_n x\sinh(\lambda_n y)$

最後的邊界條件$u(x,b)=1$代入，得

$1=\frac{F_0}{2}+\sum_{n=1}^{\infty}F_n\cos\lambda_n x\sinh(\lambda_n b)\Rightarrow$知所有$F_n=0, n=1,2,3,\cdots$，而$F_0=2$

$u(x,y)=\frac{y}{b}$